GENERAL FIBONACCI SEQUENCES IN FINITE GROUPS

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(Submitted August 1996–Final Revision June 1997)

1. INTRODUCTION

The study of Fibonacci sequences in groups began with the earlier work of Wall [7], where the ordinary Fibonacci sequences in cyclic groups were investigated. Another early contributor to this field was Vinson , who was particularly interested in ranks of apparition in ordinary Fibonacci sequences [6]. In the mid eighties, Wilcox extended the problem to abelian groups [8]. Campbell, Doostie, and Robertson expanded the theory to some finite simple groups [2]. One of the latest works in this area is [1], where it is shown that the lengths of ordinary 2-step Fibonacci sequences are equal to the length of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and exponent a prime number p. The theory has been generalized in [3] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups of nilpotency class 2 and exponent p.

Definition 1.1: Let $H \lhd G$, $K \lhd G$, and $K \leq H$. If H/K is contained in the center of G/K, then H/K is called a *central factor* of G. A group G is called *nilpotent* if it has finite series of normal subgroups $G = G_0 \geq G_1 \geq \cdots \geq G_r = 1$ such that G_{i-1}/G_i is a central factor of G for each i = 1, 2, ..., r. The smallest possible r is call the *nilpotency class* of G.

Further details about nilpotent groups and related topics can be found in [4].

Let G be a free nilpotent group of nilpotency class 2 and exponent p. G has a presentation $G = \langle x, y, z : x^p = 1, y^p = 1, z^p = 1, z = (y, x) = y^{-1}x^{-1}yx\rangle$. Suppose that we have integers n and m and a recurrence relation in this group given by

$$x_{i-2}^n * x_{i-1}^m = x_i \quad \forall i \in Z.$$

We assume that p does not divide n. Then we get a definition of a 2-step general standard Fibonacci sequence which will be $(0, 1, m, n+m^2, \ldots)$ in $\mathbf{Z}/\mathbf{pZ}$. If p were permitted to divide n, then the sequence ultimately would be periodic, but would never return to the consecutive pair $0, 1$. The length of the standard sequence is k, which we call the *Wall number* of the sequence, sometimes called the *fundamental period* of that sequence.

Each element in the group G can be represented uniquely as $x^a y^b z^c$, where $a, b, c \in \mathbf{Z}/\mathbf{pZ}$. The group relations give us a law of composition of standard forms

$$x^a y^b z^c \cdot x^{a'} y^{b'} z^{c'} = x^{a''} y^{b''} z^{c''},$$

where a'', b'', and c'' are given by the following explicit formulas.

We have $a'' = a + a'$, $b'' = b + b'$, and $c'' = c + c' + a'b$. These product laws are discussed in more detail in [1]. In order to study this recurrence, we need a closed formula to describe how to take the next term of the sequence. Let $(x^a y^b z^c)^n$ and $(x^{a'} y^{b'} z^{c'})^m$ be two elements in G. The relevant formulas are

$$(x^a y^b z^c)^n (x^{a'} y^{b'} z^{c'})^m = x^{a''} y^{b''} z^{c''},$$

where

$$a'' = na + ma',$$
$$b'' = nb + mb',$$

and

$$c'' = nc + mc' + mna'b + \frac{(n-1)n}{2}ab + \frac{(m-1)m}{2}a'b'.$$

2. THE MAIN RESULT AND PROOF

Let us use vector notation to calculate the sequence. We put $(1,0,0) = (s_{-1}, r_0, t_0)$ which corresponds to x, and $(0,1,0) = (s_0, r_1, t_1)$ which corresponds to y. We demonstrate more vectors using the above product formula for c'' as

$$(n, m, 0) = (s_1, r_2, t_2) \text{ and } \left(mn, m^2+n, mn^2 + \binom{m}{2}mn\right) = (s_2, r_3, t_3).$$

We obtain two sequences (r_i) and (t_i) via our recurrence. Notice that we have $s_i = nr_i$ for each integer i. By induction on j, the j^{th} term of the third component of our sequence of vectors is

$$t_j = mn^2 \sum_{i=0}^{j-1} r_{j-i-1} r_i^2 + \binom{n}{2} n \sum_{i=0}^{j-1} r_{j-i-1} r_i r_{i-1} + \binom{m}{2} n \sum_{i=0}^{j-1} r_{j-i-1} r_i r_{i+1}.$$

Let us denote the period of the general Fibonacci sequence in the group G by $k(G)$.

Theorem 2.1: Let $p > 3$ be a prime number. Then, if G is a nontrivial finite p-group of exponent p and nilpotency class 2, $k(G) = k$. There are four assumptions that we will insert:

a) $n \not\equiv 0 \pmod{p}$,

b) $m + n - 1 \not\equiv 0 \pmod{p}$,

c) $n^2 - m^3 - n - 3mn \not\equiv 0 \pmod{p}$,

d) $3m(m^2+n) \not\equiv 0 \pmod{p}$.

Proof: Let

$$t_k = mn^2 \sum_{i=0}^{k-1} r_{k-i-1} r_i^2 + \binom{n}{2} n \sum_{i=0}^{k-1} r_{k-i-1} r_i r_{i-1} + \binom{m}{2} n \sum_{i=0}^{k-1} r_{k-i-1} r_i r_{i+1},$$

where $m, n \in \mathbf{Z}/\mathbf{pZ}$, $p > 2$. In order to show $k(G) = k$, we must check that $t_k = t_{k+1} = 0$. The range of all the following sums is the same as above. Since $r_{i+1} = mr_i + nr_{i-1}$, we can recast the last sum to obtain

$$t_k = \left(mn^2 + \binom{m}{2}mn\right) \sum r_{k-i-1} r_i^2 + \left(\binom{n}{2}n + \binom{m}{2}n^2\right) \sum r_{k-i-1} r_i r_{i-1}.$$

We separate this sum to the two parts,

$$\theta_1 = \left(mn^2 + \binom{m}{2}mn\right) \sum r_{k-i-1} r_i^2 \quad \text{and} \quad \theta_2 = \left(\binom{n}{2}n + \binom{m}{2}n^2\right) \sum r_{k-i-1} r_i r_{i-1}.$$

We can pull out factors without difficulty. We put

$$l_1 = mn^2 + \binom{m}{2}mn \quad \text{and} \quad l_2 = \binom{n}{2}n + \binom{m}{2}n^2$$

and then set

$$\phi_1 = \sum r_{k-i-1}r_i^2 \quad \text{and} \quad \phi_2 = \sum r_{k-i-1}r_ir_{i-1}.$$

Now we have $\theta_1 = l_1\phi_1$ and $\theta_2 = l_2\phi_2$, and we are in a position to show that $\phi_1 = 0$ and $\phi_2 = 0$. First, we prove that

$$\phi_2 = \sum r_{k-i-1}r_ir_{i-1} = \sum r_{-(i+1)}r_ir_{i-1} = 0.$$

Now let us show that

$$r_{-i} = (-1)^{i+1}\left(\frac{1}{n}\right)^i r_i.$$

If α and β are the roots of $x^2 - mx - n = 0$, then $\alpha\beta = -n$ and $\alpha + \beta = m$. We have, from the Binet formula,

$$r_i = \frac{\alpha^i - \beta^i}{\alpha - \beta} \quad \text{and} \quad r_{-i} = \frac{\alpha^{-i} - \beta^{-i}}{\alpha - \beta}.$$

We multiply r_{-i} by $(\alpha\beta)^i$ to see that

$$r_{-i} = (-1)^{i+1}\left(\frac{1}{n}\right)^i r_i, \tag{1}$$

and also we have

$$r_{i+1}r_{i-1} = r_i^2 - (-n)^{i-1}. \tag{2}$$

This formula was known to Somer [5]. By using $r_{-(i+1)} = (-1)^i(\frac{1}{n})^{i+1}r_{i+1}$ and (2), we obtain

$$\sum r_{-(i+1)}r_ir_{i-1} = \sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_i^3 + \frac{1}{n^2}\sum r_i.$$

We will prove that $\sum r_i = 0$. Since our recurrence relation is $r_i = mr_{i-1} + nr_{i-2}$, we deduce that $\sum r_i = m\sum r_{i-1} + n\sum r_{i-2}$. Replace $i-1$ by i in the first sum and $i-2$ by i in the second sum on the right side to yield·

$$(m+n-1)\sum r_i = 0. \tag{3}$$

Thus, $\sum r_i = 0$ unless $m+n-1$ is congruent to 0 modulo p. The next step is to show that

$$\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_i^3 = 0,$$

so we will be half way through the proof. From the recurrence relation,

$$\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_i^3 = \sum (-1)^i\left(\frac{1}{n}\right)^{i+1} (mr_{i-1} + nr_{i-2})^3.$$

We expand this equation to obtain

$$\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_i^3 = m^3\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i-1}^3 + 3m^2n\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i-1}^2r_{i-2}$$
$$+ 3mn^2\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i-1}r_{i-2}^2 + n^3\sum (-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i-2}^3.$$

Replacing $i-1$ by i in the first, second, and third sums, and $i-2$ by i in the last sum on the right side, we obtain

$$\begin{aligned}\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 = m^3\sum(-1)^{i+1}\left(\frac{1}{n}\right)^{i+2}r_i^3 + 3m^2n\sum(-1)^{i+1}\left(\frac{1}{n}\right)^{i+2}r_i^2r_{i-1} \\ + 3mn^2\sum(-1)^{i+1}\left(\frac{1}{n}\right)^{i+2}r_ir_{i-1}^2 + n^3\sum(-1)^{i+2}\left(\frac{1}{n}\right)^{i+3}r_i^3.\end{aligned} \tag{4}$$

Now we have

$$\left(n-\frac{m^3}{n}-1\right)\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 + 3mn\sum(-1)^{i+1}\left(\frac{1}{n}\right)^{i+2}r_ir_{i-1}(mr_i+nr_{i-1}) = 0.$$

Using $mr_i+nr_{i-1}=r_{i+1}$ and $r_{i+1}r_{i-1}=r_i^2-(-n)^{i-1}=r_i^2+(-1)^i(n)^{i-1}$, we obtain

$$\left(n-\frac{m^3}{n}-1\right)\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 - 3m\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 - 3m\sum\frac{1}{n^2}r_i = 0.$$

The last sum is zero by (3). Then we have

$$\left(n-\frac{m^3}{n}-1-3m\right)\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 = 0. \tag{5}$$

We multiply (5) by n to see that

$$(n^2-m^3-n-3mn)\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 = 0.$$

Finally, we have

$$\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 = 0, \tag{6}$$

unless $n^2-m^3-n-3mn$ is congruent to 0 modulo p. We deduce that $\phi_2=0$. Hence, we have completed the first part of the proof. Now we prove that the other part of t_k is 0. By (1), write

$$\phi_1 = \sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_{i+1}r_i^2.$$

By (4), we have

$$\begin{aligned}(n^2-m^3-n)\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 + 3m^2n^2\sum(-1)^{i+1}\left(\frac{1}{n}\right)^{i+2}r_i^2r_{i-1} \\ + 3mn^3\sum(-1)^{i+1}\left(\frac{1}{n}\right)^{i+2}r_ir_{i-1}^2 = 0.\end{aligned}$$

From (6), we have our first linear equation:

$$3m^2n\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^2r_{i-1} + 3mn^2\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_ir_{i-1}^2 = 0. \tag{7}$$

Therefore, from the recurrence relation $nr_i = r_{i+2}-mr_{i+1}$ and (6), we get

$$\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}r_i^3 = \frac{1}{n^3}\sum(-1)^i\left(\frac{1}{n}\right)^{i+1}(r_{i+2}-mr_{i+1})^3 = 0.$$

We exploit this equation to obtain

$$\frac{1}{n^3}\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i+2}^3 - 3\frac{m}{n^3}\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i+2}^2 r_{i+1} + 3\frac{m^2}{n^3}\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i+2} r_{i+1}^2$$
$$-\frac{m^3}{n^3}\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i+1}^3 = 0.$$

Replace $i+2$ by i in the first, second, and third sums and $i+1$ by i in the last sum on the left side to see that

$$\frac{1}{n^3}\sum(-1)^{i-2}\left(\frac{1}{n}\right)^{i-1} r_i^3 - 3\frac{m}{n^3}\sum(-1)^{i-2}\left(\frac{1}{n}\right)^{i-1} r_i^2 r_{i-1} + 3\frac{m^2}{n^3}\sum(-1)^{i-2}\left(\frac{1}{n}\right)^{i-1} r_i r_{i-1}^2$$
$$-\frac{m^3}{n^3}\sum(-1)^{i-1}\left(\frac{1}{n}\right)^{i} r_i^3 = 0.$$

The first and last sums vanish by (6). We multiply the equation by n to obtain a second linear equation

$$-3m\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_i^2 r_{i-1} + 3m^2\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_i r_{i-1}^2 = 0. \tag{8}$$

Hence, from the linear equations (7) and (8),

$$\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_i^2 r_{i-1} = 0 \tag{9}$$

and

$$\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_i r_{i-1}^2 = 0, \tag{10}$$

unless $3mn(m^2+n)$ is congruent to 0 modulo p. Replacing $i-1$ by i in (10),

$$3m(m^2+n)\sum(-1)^i\left(\frac{1}{n}\right)^{i+1} r_{i+1} r_i^2 = 0.$$

So we have finished the second part of the proof. Therefore, we have $t_k = 0$.

Similarly,

$$t_{k+1} = mn^2\sum_{i=0}^{k} r_{k-i} r_i^2 + \binom{n}{2} n\sum_{i=0}^{k} r_{k-i} r_i r_{i-1} + \binom{m}{2} n\sum_{i=0}^{k} r_{k-i} r_i r_{i+1}.$$

From (1), we have

$$t_{k+1} = mn^2\sum_{i=0}^{k}(-1)^{i+1}\left(\frac{1}{n}\right)^i r_i^3 + \binom{n}{2} n\sum_{i=0}^{k}(-1)^{i+1}\left(\frac{1}{n}\right)^i r_i^2 r_{i-1} + \binom{m}{2} n\sum_{i=0}^{k}(-1)^{i+1}\left(\frac{1}{n}\right)^i r_i^2 r_{i+1}$$

This is the same as

$$t_{k+1} = mn^2\sum_{i=0}^{k-1}(-1)^{i+1}\left(\frac{1}{n}\right)^i r_i^3 + \binom{n}{2} n\sum_{i=0}^{k-1}(-1)^{i+1}\left(\frac{1}{n}\right)^i r_i^2 r_{i-1} + \binom{m}{2} n\sum_{i=0}^{k-1}(-1)^{i+1}\left(\frac{1}{n}\right)^i r_i^2 r_{i+1}$$
$$+ mn^2(-1)^{k+1}\left(\frac{1}{n}\right)^k r_k^3 + \binom{n}{2} n(-1)^{k+1}\left(\frac{1}{n}\right)^k r_k^2 r_{k-1} + \binom{m}{2} n(-1)^{k+1}\left(\frac{1}{n}\right)^k r_k^2 r_{k+1}.$$

The last three terms are zero by the fact that $r_k = 0$ because the period of the sequence r_i is k. The first three sums are zero by exactly the same argument as in the proof of $t_k = 0$. Hence, $t_{k+1} = 0$. To be more explicit, the same restrictions are still valid for $t_{k+1} = 0$. Thus, the proof of Theorem 2.1 is completed.

This result has an obvious interpretation in terms of quotients of groups with presentations similar to those of Fibonacci groups, which is

$$F(2, r, m, n) = \langle x_1, x_2, \ldots, x_r : x_1^n x_2^m x_3^{-1} = 1, x_2^n x_3^m x_4^{-1} = 1, \ldots, x_{r-1}^n x_r^m x_1^{-1} = 1, x_r^n x_1^m x_2^{-1} = 1 \rangle.$$

ACKNOWLEDGMENT

The authors would like to express their appreciation to the anonymous referee for improving the statement and proof of Theorem 2.1 and for making useful suggestions regarding the presentation of this paper.

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AMS Classification Number: 11B39