CONJECTURES ON THE Z-DENSITIES OF THE FIBONACCI SEQUENCE

Paul S. Bruckman

13 Webster Ave., Apt. G., Highwood, IL 60040

Peter G. Anderson

Dept. of Computer Science, Rochester Institute of Technology, Rochester, NY 14623-5608

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1. INTRODUCTION

The concept of "Z-densities" is introduced in this paper, leading to several interesting conjectures involving the divisibility properties of the Fibonacci entry-point function.

We let $\mathscr{F} = \{F_n\}_{n=1}^{\infty}$ and $\mathscr{L} = \{L_n\}_{n=1}^{\infty}$ denote the Fibonacci and Lucas sequences, respectively. Given m, the *Fibonacci entry-point of m*, denoted by $Z(m)$, is the smallest $n > 0$ such that $m|F_n$; in this case, we write $Z(m) = n$. If m and n are arbitrary (with $m > 1$), $m|F_n$ iff $Z(m)|n$.

If $m = p$, a prime $\neq 2$ or 5, it is well-known that $Z(p)|(p-\varepsilon_p)$, where $\varepsilon_p = (5/p)$, the Legendre symbol.

Given an arbitrary sequence $\mathscr{U} = \{U_n\}$ of positive integers, we say that p *divides* $\mathscr{U}$, and write $p|\mathscr{U}$ iff $p|U_n$ for some n. Let $\pi_{\mathscr{U}}(x)$ denote the number of primes $p \le x$ such that $p|\mathscr{U}$; also, $\pi(x)$ is the number of primes $p \le x$. The "natural" *density*, or simply the *density*, of $\mathscr{U}$ is given by

$$\theta_{\mathscr{U}} = \lim_{x\to\infty} \pi_{\mathscr{U}}(x) / \pi(x). \tag{1.1}$$

It is well-known that $p|\mathscr{F}$ for all p, and so $\theta_{\mathscr{F}} = 1$. This is certainly not the case for a general $\mathscr{U}$. J. C. Lagarias [6] has determined $\theta_{\mathscr{U}}$ for a few specific sequences, among them $\mathscr{L}$. As far as the topic of this paper is concerned, the most interesting result obtained by Lagarias was the following:

$$\theta_{\mathscr{L}} = 2/3. \tag{1.2}$$

That is to say, $2/3$ of all primes, asymptotically, divide some Lucas number.

Now, it is also known that $p|\mathscr{L}$ iff $Z(p)$ is even. It follows that the density of those primes p for which $Z(p)$ is even is equal to $2/3$; note that this extends our initial definition of "density." The aim of the present paper is to generalize this perspective. Thus, we ask the question: Given m, what is the density of those p for which $m|Z(p)$?

We can also ask the more fundamental question: Given m, what is the density of those p such that $Z(p) = m$? However, it is clear that such densities are zero for all m, since they characterize the *primitive prime divisors* of F_m (for a given m), which are necessarily finite in number; therefore, the density of those p such that $Z(p) = m$ is of no interest to us here.

To obtain answers to the first question above, we introduce various types of densities that involve $Z(p)$ in their definitions; such densities are referred to as "Z-densities." Here is a formal definition: Given m and x, let $M(m, x)$ denote the number of $p \le x$ such that $m|Z(p)$. Then we define $\zeta(m)$, the "Z-density of m as a divisor," as follows (assuming the limit exists):

$$\zeta(m) = \lim_{x\to\infty} M(m, x) / \pi(x). \tag{1.3}$$

Clearly, $\zeta(1) = 1$; also, using Lagarias' result, $\zeta(2) = \theta_{\mathscr{L}} = 2/3$.

Based on an examination of certain Fibonacci entry-point data [4], [5], one of the authors (Bruckman) reached some conclusions regarding the evaluation of $\zeta(m)$ and related Z-densities. More recently, the other author (Anderson) has strengthened the evidence for these conjectures, using extended data produced by computer runs. Much of the numerical evidence for the various conjectures made in this paper has been omitted in the interest of brevity. However, for the sake of demonstration, we have included in the Appendix one of the tables that comprise such evidence (in abridged form). Additional details may be obtained from either author upon request. Known or proven results are annotated in the usual manner. Conjectures and consequences of such conjectures are marked with an asterisk; in the narrative, these are referred to frequently as "conditional results," meaning "results conditional on the conjectures."

The following is one of the consequences of these conjectures, valid for all primes q:

$$\zeta(q) = q/(q^2 - 1). \tag{1.4*}$$

The characteristic polynomial of the sequences $\mathscr{F}$ and $\mathscr{L}$ has the irrational zeros α and β, the familiar Fibonacci constants. For sequences having a second-degree characteristic polynomial that has *integral* zeros, (1.4)* was proved by C. Ballot [1]. Thus, Ballot's result is, conditionally, more broadly applicable. The methods employed by Ballot to establish his result are beyond the scope of this exploratory paper.

In the present work, the authors have restricted their analysis to the sequences $\mathscr{F}$ and $\mathscr{L}$. Further generalizations are left to other researchers.

Before proceeding to the main points of this paper, we find it convenient to decompose the appropriate Z-densities into certain "component" Z-densities, defined below. Our study of such component Z-densities led to the main conjectures we formulated.

In this paper, lower-case letters represent nonnegative integers, except for x, which may be any positive real number (generally thought of as large). However, the letters m and n represent positive integers, and the letters p and q represent primes.

2. COMPONENT Z-DENSITIES

We begin with a basic definition of "q^i, q^j Z-densities." Given q, x, i, and j, with $i \ge j \ge 0$, let $M(q, x; i, j)$ denote the number of $p \le x$ such that $q^i \| (p - \varepsilon_p)$ and $q^j \| Z(p)$. The expression $q^0 \| n$ is taken to mean $q \nmid n$. Then the "q^i, q^j Z-density," denoted $\zeta(q; i, j)$, is given as follows (assuming the limit exists):

$$\zeta(q; i, j) \equiv \lim_{x \to \infty} M(q, x; i, j) / \pi(x). \tag{2.1}$$

On the basis of empirical evidence, we formulate the following conjecture.

Conjecture 2.1*:

$$\zeta(q; i, j) = \begin{cases} (q-2)/(q-1) & \text{if } i = j = 0, \\ q^{-2i} & \text{if } i \ge 1, j = 0, \\ (q-1)q^{-1-2i+j} & \text{if } i \ge j \ge 1. \end{cases}$$

By the definition of $\zeta(q; i, j)$, it is clear that, for all primes q:

$$\sum_{i \ge j \ge 0} \zeta(q; i, j) = 1. \tag{2.2}$$

It is readily verified that Conjecture 2.1* implies (2.2).

Conjecture 2.1* appears to hold even for the "exceptional" primes 2 and 5, which play a special role in the study of $\mathcal{F}$ and $\mathcal{L}$. However, a different type of rule applies when we study the divisibility of $Z(p)$ by both 2 and 5 *in conjunction*. This rule is considerably more complex than that indicated in Conjecture 2.1*, must be offered in the form of a (two-dimensional) table, and requires a special definition:

Given x, i, j, k, and l, with $i \ge j \ge 0$, $k \ge l \ge 0$, we let $M(2, 5, x; i, j; k, l)$ denote the number of $p \le x$ such that $2^i 5^k \| (p - \varepsilon_p)$ and $2^j 5^l \| Z(p)$. Then the "$2^i, 2^j, 5^k, 5^l$ Z-density," denoted $\zeta(2, 5; i, j; k, l)$, is defined as follows:

$$\zeta(2, 5; i, j; k, l) \equiv \lim_{x \to \infty} M(2, 5, x; i, j; k, l) / \pi(x). \tag{2.3}$$

The numerical evidence, combined with general reasoning, suggests the following conjecture.

Conjecture 2.2*: $\zeta(2, 5; i, j; k, l)$.

$(i, j) \setminus (k, l)$:	$(0, 0)$	$k \ge 1$, $l = 0$	$k \ge l \ge 1$	Row Totals
$(0, 0)$	0	0	0	0
$(1, 0)$	$1/4$	0	0	$1/4$
$(1, 1)$	$1/8$	$1/2 \cdot 5^{-2k}$	$2 \cdot 5^{-1-2k+l}$	$1/4$
$i \ge 2, j = 0$	2^{-1-2i}	$2^{1-2i}5^{-2k}$	$2^{3-2i}5^{-1-2k+l}$	2^{-2i}
$(i, i), i \ge 2$	2^{-1-i}	0	0	2^{-1-i}
$i > j \ge 2$	$2^{-2-2i+j}$	$2^{-2i+j}5^{-2k}$	$2^{2-2i+j}5^{-1-2k+l}$	$2^{-1-2i+j}$
Column Totals	$3/4$	5^{-2k}	$4 \cdot 5^{-1-2k+l}$	1

The row totals in Conjecture 2.2* are the sums over all $k \ge l \ge 0$ and are the $\zeta(2; i, j)$ obtained by setting $q = 2$ in Conjecture 2.1*. Likewise, the column totals are the sums over all $i \ge j \ge 0$ and are the $\zeta(5; k, l)$ obtained from Conjecture 2.1* by setting $q = 5$ and replacing (i, j) by (k, l). Therefore, our conjectures are mutually consistent.

The Z-densities introduced above give information about the divisibility properties of $(p - \varepsilon_p)$ and $Z(p)$. We now derive expressions for Z-densities that only yield information about the divisibility properties of $Z(p)$. Accordingly, we make the following definitions:

$$\zeta(q; j) \equiv \sum_{r \ge 0} \zeta(q; r + j, j); \tag{2.4}$$

$$\zeta(2, 5; j, l) \equiv \sum_{r \ge 0} \sum_{s \ge 0} \zeta(2, 5; r + j, j; s + l, l). \tag{2.5}$$

Note that $\zeta(q; j)$ is the density of those primes p for which $q^j \| Z(p)$, and $\zeta(2, 5; j, l)$ is the density of those p for which $2^j 5^l \| Z(p)$. If we substitute the putative results from Conjectures 2.1* and 2.2*, respectively, into the formulas indicated in (2.4) and (2.5), we obtain the following expressions:

$$\zeta(q;j)=\begin{cases}(q^2-q-1)/(q^2-1) & \text{if } j=0,\\ q^{1-j}/(q+1) & \text{if } j\geq 1.\end{cases} \tag{2.6*}$$

TABLE 2.1*. $\zeta(2,5;j,l)$

	$l=0$	$l\geq 1$	Row Totals
$j=0$	$43/144$	$5^{1-l}/36$	$1/3$
$j=1$	$7/36$	$5^{1-l}/9$	$1/3$
$j\geq 2$	$43\cdot 2^{-j}/72$	$2^{-j}5^{1-l}/18$	$2^{1-j}/3$
Column Totals	$19/24$	$5^{1-l}/6$	1

The row totals in Table 2.1* coincide with the $\zeta(2;j)$ from (2.6)*; the column totals coincide with the $\zeta(5;l)$ from (2.6)* (obtained by setting $q=5$ and replacing j by l).

We next require an additional set of Z-densities, this time involving mere divisibility of $Z(p)$ by q^j, instead of *exact* divisibility. Note that $q^j \mid Z(p)$ iff there exists some $r\geq 0$ such that $q^{r+j}\,\|\,Z(p)$. Since r satisfying this condition is arbitrary, this suggests the following relations:

$$\zeta(q^j)=\sum_{r\geq 0}\zeta(q;r+j); \tag{2.7}$$

$$\zeta(2^j s^l)=\sum_{r\geq 0}\sum_{s\geq 0}\zeta(2,5;r+j,s+l). \tag{2.8}$$

The density $\zeta(2^j5^l)$, according to definition (1.3), is the density of those p such that $2^j5^l \mid Z(p)$.

Substituting the conditional results from (2.6)* and Table 2.1* into the expressions in (2.7) and (2.8), we obtain the following:

$$\zeta(q^j)=\begin{cases}1 & \text{if } j=0,\\ q^{2-j}/(q^2-1) & \text{if } j\geq 1;\end{cases} \tag{2.9*}$$

$$\zeta(2^j5^l)=\begin{cases}1 & \text{if } j=l=0,\\ 5^{2-l}/24 & \text{if } j=0, l\geq 1,\\ 5^{3-l}/144 & \text{if } j=1, l\geq 1,\\ 2^{2-j}/3 & \text{if } j\geq 1, l=0,\\ 2^{-j}5^{2-l}/36 & \text{if } j\geq 2, l\geq 1.\end{cases} \tag{2.10*}$$

Note that if we set $l=0$ in (2.10)*, we obtain $\zeta(2^j)$ as indicated from (2.9)* with $q=2$; likewise, setting $j=0$ in (2.10)* yields $\zeta(5^l)$, obtained from (2.9)* by setting $q=5$ and replacing j by l. Such numerical checks inspire confidence in the validity of our conjectures.

In the next section we use the conditional results obtained in this section to derive a general expression for $\zeta(m)$.

3. DERIVATION OF $\zeta(m)$

We would normally expect that the Z-densities satisfy a multiplicative property of sorts; naively, we might suppose that $\zeta(m) = \prod_{q^j \| m} \zeta(q^j)$. However, there is apparently a certain amount of "distortion" in this putative multiplicativity law, due to the presence of the "special" densities $\zeta(2^j 5^l)$ that might enter into the computation. In order to measure this distortion, we introduce a ratio defined as follows:

$$\rho(j, l) \equiv \zeta(2^j 5^l) / (\zeta(2^j)\zeta(5^l)). \tag{3.1}$$

Computing $\rho(j, l)$ from (2.9)* and (2.10)* is a relatively simple matter, and we obtain the following expressions:

$$\rho(j, l) = \begin{cases} 1/2 & \text{if } j \geq 2, l \geq 1, \\ 5/4 & \text{if } j = 1, l \geq 1, \\ 1 & \text{if } j = 0 \text{ or } l = 0. \end{cases} \qquad (3.2)^*$$

Based on the foregoing comments, we postulate the following "quasi-multiplicative" property.

Conjecture 3.1*:

$$\zeta(m) = \rho(j, l) \prod_{q^e \| m} \zeta(q^e), \text{ whenever } 2^j 5^l \| m.$$

We may also redefine $\rho(j, l)$ as an explicit function of m, as follows:

$$\rho(m) = \begin{cases} 1 & \text{if } 10 \nmid m, \\ 5/4 & \text{if } m \equiv 10 \pmod{20}, \\ 1/2 & \text{if } 20 \mid m. \end{cases} \qquad (3.3)^*$$

Therefore, our quasi-multiplicative property now takes the following form:

$$\zeta(m) = \rho(m) \prod_{q^j \| m} \zeta(q^j). \qquad (3.4)^*$$

We may now substitute the values of $\zeta(q^j)$ from (2.9)* into the formula given by (3.4)*. Note the following:

$$\zeta(m) / \rho(m) = \prod_{q^j \| m} q^{2-j} / (q^2 - 1) = t(m) / m, \text{ where}$$

$$t(m) \equiv \prod_{q \mid m} (1 - q^{-2})^{-1}, \; m > 1; \; t(1) = 1. \tag{3.5}$$

Therefore, we obtain our final formula for $\zeta(m)$:

$$\zeta(m) = \rho(m) t(m) / m, \qquad (3.6)^*$$

where $\rho(m)$ and $t(m)$ are given by (3.3)* and (3.5), respectively. As we may verify, this formula yields the known results: $\zeta(1) = 1$ and $\zeta(2) = 2/3$. Additional (conditional) results yielded by the general formula in (3.6)* are as follows: $\zeta(3)=3/8$, $\zeta(4)=1/3$, $\zeta(5)=5/24$, etc. The conditional result of (3.6)*, if true, implies that $\zeta(m)$ is rational (and positive) for all m.

Conditionally, the function $\zeta(m)$ is not multiplicative, while the function $\zeta(m)/\rho(m)$ *is*. Thus, if m and n are coprime, we have the interesting property

$$\zeta(mn)=\rho(mn)/(\rho(m)\rho(n))\cdot\zeta(m)\zeta(n). \tag{3.7}*$$

Other interesting derived conditional properties of $\zeta(m)$ follow from (3.6)*. In the interest of brevity, we omit the demonstration of these properties, and merely indicate the results. For example, (3.6)* implies the following:

$$\sum_{m\geq 1}\zeta(m)/m=\frac{5\cdot 7\cdot 11\cdot 31{,}277}{2^4\cdot 601\cdot 691}=12{,}041{,}645/6{,}644{,}656. \tag{3.8}*$$

We may also show (conditionally) that the average order of $\zeta(m)$, over all $m\leq x$, if $O(\log x/x)$, but omit the demonstration.

We have omitted discussion of the densities of those p for which $Z(p)=m$, where m is a specified positive integer. Such a density relates to the number of *primitive prime divisors* (p.p.d.'s) of F_m, since these are precisely those primes p such that $Z(p)=m$. Hence, this density must be zero for all values of m, since the number of p.p.d.'s of F_m must be finite. On the other hand, the principles previously employed lead to a formula for such density in terms of the component densities obtained in Section 2. Proceeding thus, we find that each such resultant density has a "constant" multiplier denoted as δ, where

$$\delta\equiv\prod_p\{(p^2-p-1)/(p^2-1)\}. \tag{3.9}$$

However, the infinite product defining such "constant" δ diverges to zero. To see this, note that

$$0\leq\delta=\prod_p\{(1-p/(p^2-1)\}<\prod_p\{1-1/p\};$$

since it is well known that the latter product is divergent to zero, we see that $\delta=0$. This, in turn, implies that the density of those primes p such that $Z(p)=m$, as anticipated.

From the definition of density and the Prime Number Theorem, we deduce that, for a given m, the number of p.p.d.'s of F_m is $o(x/\log x)$ for all $m\leq x$. In fact, it seems probable that the number of p.p.d.'s of F_m is $O(\log x)$, which is certainly $o(x/\log x)$. The conditional demonstration of this last statement is deferred, as it will be the subject of a future paper.

4. NUMERICAL VERIFICATION

In the interest of brevity, we have omitted all but one of the appendices that originally formed part of this paper. These contain the results of certain statistical tests conducted by the authors to test the validity of the conjectures. The tests were conducted by analyzing the data on $Z(p)$ and $p-\varepsilon_p$ for the first million primes (the highest such prime being 15,485,863). Although due caution is required in conducting any such tests, *if* we accept their validity, it may be stated with better than 95% statistical confidence that the conjectures are correct.

For the sake of demonstration, we have included one of these tests (in abridged form) in Appendix 1. Anyone interested in seeing the complete results of such analysis may contact either author for copies thereof.

The numerical evidence based on these studies supports our belief that the underlying conjectures made in this paper are correct. However, statistical corroboration does not constitute mathematical proof, and proof is what is required to establish these conjectures rigorously.

APPENDIX 1

$x = 15,485,863$; $\pi(x) = 1,000,000$

q	j	(1) $M(q, x; j)$	(2) $\zeta(q; j)$	(3) $\pi(x) \cdot \zeta(q; j)$	(4) = $[(1)-(3)]^2 \div (3)$
2	0	333,286	.3333333	333,333	0.0066
2	1	333,329	.3333333	333,333	0.0000
2	2	166,737	.1666667	166,667	0.0294
2	3	83,216	.0833333	83,333	0.1643
2	4	41,734	.0416667	41,667	0.1077
2	5	20,896	.0208333	20,833	0.1905
2	6	10,460	.0104167	10,417	0.1775
2	7	5,185	.0052083	5,208	0.1016
2	8	2,591	.0026042	2,604	0.0649
2	9	1,307	.0013021	1,302	0.0192
2	10	626	.0006510	651	0.9601
2	11	326	.0003255	326	0.0000
2	12	152	.0001628	163	0.7423
2	13	75	.0000814	81	0.4444
2	14	47	.0000407	41	0.8780
2	15	14	.0000203	20	1.8000
2	16-21	19	.0000200	20	0.0500
Totals for q = 2:		1,000,000		999,999	5.7365
3	0	625,126	.6250000	625,000	0.0254
3	1	249,889	.2500000	250,000	0.0493
3	2	83,271	.0833333	83,333	0.0461
3	3	27,764	.0277778	27,778	0.0071
3	4	9,331	.0092593	9,259	0.5599
3	5	3,073	.0030864	3,086	0.0548
3	6	1,028	.0010288	1,029	0.0010
3	7	330	.0003429	343	0.4927
3	8	138	.0001143	114	5.0526
3	9	32	.0000381	38	0.9474
3	10-12	18	.0000183	18	0.0000
Totals for q = 3:		1,000,000		999,998	7.2363

APPENDIX 1 (continued)

q	j	(1) $M(q, x; j)$	(2) $\zeta(q; j)$	(3) $\pi(x) \cdot \zeta(q; j)$	(4) = $[(1)-(3)]^2 \div (3)$
5	0	791,679	.7916667	791,667	0.0002
5	1	166,700	.1666667	166,667	0.0065
5	2	33,272	.0333333	33,333	0.1116
5	3	6,612	.0066667	6,667	0.4537
5	4	1,396	.0013333	1,333	2.9775
5	5	278	.0002667	267	0.4532
5	6	51	.0000533	53	0.0755
5	7-8	12	.0000128	13	0.0769
Totals for q = 5:		1,000,000		1,000,000	4.1551
7	0	854,407	.8541667	854,167	0.0674
7	1	124,742	.1250000	125,000	0.5325
7	2	17,907	.0178571	17,857	0.1400
7	3	2,533	.0025510	2,551	0.1270
7	4	356	.0003644	364	0.1758
7	5-7	55	.0000606	61	0.5902
Totals for q = 7:		1,000,000		1,000,000	1.6329
11	0	908,281	.9083333	908,333	0.0030
11	1	83,400	.0833333	83,333	0.0539
11	2	7,581	.0075758	7,576	0.0033
11	3	676	.0006887	689	0.2453
11	4-5	62	.0000683	68	0.5294
Totals for q = 11:		1,000,000		1,000,000	0.8349

SUMMARY

Grouped Values of q	Number of Values	Total Number of Data Points (n)	Chi-Square Statistic	χ^2 Value at 97.5% Confidence
2, 3	2	28	12.9728	14.5733
5,7,11	3	19	6.6229	8.2308
2,3,5,7,11	5	47	19.5957	27.60 (est.)

APPENDIX 1-SUMMARY (continued)

Explanation:

1. $M(q, x; j) = \Sigma_{i\geq 0} M(q, x; i+j, j)$ enumerates those primes $p \leq x$ such that, for the given prime q, $q^j \| Z(p)$.
2. Column (2) is obtained from the formula given in (2.6)*.
3. In the last data point for each q, values of $M(q, x; j)$ were aggregated with preceding values, in some cases, so as to make the aggregated value 12 or more. This was done to minimize the distortion in the calculated value of the Chi Square statistic. For these entries, Columns (2) and (3) reflect the sum of the values for the indicated values of j.
4. The values of χ^2 at the 97.5% confidence level are taken from *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ed. M. Abramowitz & I. A. Stegun (National Bureau of Standards, 9th ptg., 1970). These values are read using $n-1$ as the degrees of freedom.
5. In this Summary, the Chi Square statistic is less than the corresponding χ^2 value at the 97.5% confidence level. This latter amount is the value at which the "tail" of the distribution function, for the indicated degrees of freedom, is .025. Therefore, on the basis of this test alone, we would accept the conjecture in (2.6)* involving $q = 2, 3, 5, 7$, or 11, with 97.5% confidence.

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REFERENCES

1. C. Ballot. "Density of Prime Divisors of Linear Recurrences." *Memoirs of the Amer. Math. Soc.* **115.551** (1995) (Providence, RI.)
2. J. Brillhart, P. L. Montgomery, & R. D. Silverman. "Tables of Fibonacci and Lucas Factorizations." *Math. of Comp.* **50.181** (1988):251-60.
3. Bro. A. Brousseau, Compiler. *Fibonacci and Related Number Theoretic Tables*. San Jose: The Fibonacci Association, 1972.
4. Bro. A. Brousseau, Ed. *Tables of Fibonacci Entry Points, Part One*. Santa Clara: The Fibonacci Association, 1965.
5. Bro. A. Brousseau, Ed. *Tables of Fibonacci Entry Points, Part Two*. Santa Clara: The Fibonacci Association, 1965.
6. J. C. Lagarias. "The Set of Primes Dividing the Lucas Numbers Has Density 2/3." *Pacific J. Math.* **118.2** (1985):449-61.
7. E. Lucas. "Théorie des fonctions numeriqués simplement périodiques." *Amer. J. Math.* **1** (1898):184-240, 289-321.

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