A REMARK ABOUT THE BINOMIAL TRANSFORM

Massimo Galuzzi
Dipartimento di Matematica, Università di Milano, Italy
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In [1, p. 137], Knuth introduced the idea of the *binomial transform*.*

Given a sequence of numbers $\langle a_n \rangle$, its binomial transform $\langle \hat{a}_n \rangle$ may be defined by the rule

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \tag{1}$$

Denoting the respective generating functions of $\langle a_n \rangle$ and $\langle \hat{a}_n \rangle$ by $A(x)$ and $\hat{A}(x)$, relation (1) corresponds to

$$\hat{A}(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right). \tag{2}$$

In [2], Prodinger gives the following generalization. A sequence $\langle a_n \rangle$ may be transformed into $\langle \hat{a}_n \rangle$ by the rule

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} c^k \cdot a_k, \tag{3}$$

which corresponds to

$$\hat{A}(x) = \frac{1}{1-bx} A\left(\frac{cx}{1-bx}\right). \tag{4}$$

Now we may look at equation (4) as the action of a group structure over the set of functions.

Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{C}^*$ denote the set of complex numbers different from 0. We define a group structure in $\mathbb{C} \times \mathbb{C}^*$ by the law

$$(b, c) \circ (b', c') = (b' + bc', cc'). \tag{5}$$

Now

$$\begin{aligned} &\frac{1}{1-b'x} \frac{1}{1-b\dfrac{c'x}{1-b'x}} A\left(c\frac{\dfrac{c'x}{1-b'x}}{1-b\dfrac{c'x}{1-b'x}}\right) \\ &= \frac{1}{1-(b'+bc')x} A\left(\frac{cc'x}{1-(b'+bc')x}\right). \end{aligned} \tag{6}$$

Relation (6) shows that the action of the element (b, c) on $A(x)$ followed by the action of the element (b', c') corresponds to the action of the product $(b'+bc', cc')$ over $A(x)$.

It is easy to verify that the operation in (5) is associative. The unit element is given by $(0, 1)$, and the inverse of the element (b, c) is given by $(-b/c, 1/c)$. We immediately deduce that the inversion formula for the binomial transform is

* A slight modification has been introduced in the original definition.

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-b}{c}\right)^{n-k} \left(\frac{1}{c}\right)^{k} \cdot \hat{a}_k = c^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b^{n-k} \hat{a}_k .$$

It may be observed that the same group structure works equally well with the transformation (for d fixed),

$$A(x) \to \frac{1}{(1-bx)^d} A\left(\frac{cx}{1-bx}\right),$$

introduced by Prodinger in [2], corresponding to

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n+d-1}{n-k} b^{n-k} c^k \cdot a_k .$$

REFERENCES

1. D. E. Knuth. *The Art of Computer Programming* **3**. Reading, MA: Addison Wesley, 1973.
2. H. Prodinger. "Some Information about the Binomial Transform." *The Fibonacci Quarterly* **32.5** (1994):412-15.

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