ON A FIBONACCI RELATED SERIES

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1. INTRODUCTION

A Fibonacci-related sequence is used as motivation for the representation of a resulting infinite series in closed form. Use is made of Z transform theory in the solution of a homogeneous difference-delay equation, together with an appeal to some asymptotic properties.

2. METHOD

Consider the homogeneous difference-delay equation

$$\left.\begin{array}{ll} f_{n+1} - bf_n - cf_{n-a} = 0, & n \geq a, \\ f_{n+1} - bf_n = 0, & n < a, \end{array}\right\} \tag{1}$$

with $f_0 = 1$; a and n are positive integers including zero, and b and c are real constants.

The Z transform of a sequence $\{f_n\}$ is a function $F(z)$ of a complex variable defined by $F(z) = Z[f_n] = \sum_{n=0}^{\infty} f_n z^{-n}$ (see [6]) for those values of z for which the infinite series converges.

Taking the Z transform of equation (1) and using the initial condition $f_0 = 1$ yields, upon rearrangement,

$$F(z) = Z[f_n] = \frac{z}{z - b - cz^{-a}} = \frac{z^{a+1}}{z^{a+1} - bz^a - c}. \tag{2}$$

In particular, putting $c = b$, equation (2) may be put in the form

$$F(z) = \frac{z}{(z-b)\left[1 - \frac{bz^{-a}}{z-b}\right]},$$

and expanding in series form results in

$$F(z) = \sum_{r=0}^{\infty} \frac{b^r z^{1-ar}}{(z-b)^{1+r}}. \tag{3}$$

Convergence of the infinite series (3) is assured for $\left|\frac{bz^{-a}}{(z-b)}\right| < 1$.

The inverse Z transform of (3), from tables given in [6], is

$$f_n = \sum_{r=0}^{\infty} \binom{n-ar}{r} b^{(n-ar)} U(n-ar), \tag{4}$$

where $U(n-ar)$ is the discrete step function. Equation (4) may thus be rewritten as

$$f_n = \sum_{r=0}^{[n/(a+1)]} \binom{n-ar}{r} b^{(n-ar)}, \tag{5}$$

where $[x]$ represents the integer part of x.

The inverse Z transform of (3) may also be expressed as

$$f_n = \frac{1}{2\pi i}\int_C z^{n-1}F(z)\,dz = \sum_{j=0}^{a} z^n \mathrm{Res}_j\left(\frac{F(z)}{z}\right), \tag{6}$$

where C is a smooth Jordan curve enclosing the singularities of (2) and the integral is traversed once in an anticlockwise direction around C. [Here in (6) it may be shown that there is no contribution from the integration around the contour.]

For the restriction (which will subsequently be required for a resulting infinite series)

$$\left|\frac{(a+1)^{a+1}}{(ab)^a}\right| < 1, \tag{7}$$

the characteristic function

$$g(z) = z^{a+1} - bz^a - b \tag{8}$$

has $(a+1)$ distinct zeros ξ_j, $j = 0, 1, 2, \ldots, a$. All the singularities in (2) are therefore simple poles such that the residue, Res_j, of the poles in (2) may be evaluated as follows:

$$\mathrm{Res}_j = \lim_{z\to\xi_j}\left[(z-\xi_j)\frac{z^a}{z^{a+1}-bz^a-c}\right] = \frac{\xi_j}{(a+1)\xi_j - ab}. \tag{9}$$

From (5), and using (6) and (9), it can be concluded that

$$f_n = \sum_{r=0}^{[n/(a+1)]}\binom{n-ar}{r}b^{(n-ar)} = \sum_{j=0}^{a}\frac{\xi_j^{n+1}}{(a+1)\xi_j - ab}. \tag{10}$$

3. CONJECTURE

A Tauberian theorem [1] suggests, from (10), that

$$f_n = \sum_{r=0}^{[n/(a+1)]}\binom{n-ar}{r}b^{(n-ar)} \sim \frac{\xi_0^{n+1}}{(a+1)\xi_0 - ab}, \tag{11}$$

where ξ_0 ;is the dominant zero of (8), defined as the one with the greatest modulus.

For n large, more and more terms in the left-hand side of the series (11) are incorporated, and therefore it is *conjectured* that

$$\sum_{r=0}^{\infty}\binom{n-ar}{r}b^{(n-ar)} = \frac{\xi_0^{n+1}}{(a+1)\xi_0 - ab} \tag{12}$$

for all values of n.

Using the ratio test, the infinite series in (12) may be shown to converge in the region given by (7). A diagram of the region of convergence is shown as the shaded region of Figure 1 on the following page.

It is now worthwhile to examine briefly the location of all the zeros of (8) and highlight the fact that ξ_0, the dominant zero of (8) is always real. Details of the following statements may be seen in the work of Sofo and Cerone [4].

It may be shown, by using Rouche's theorem [5], that the characteristic function (8) with restriction (7) has exactly a zeros in the contour $\Gamma : |z| \le \left|\frac{ab}{a+1}\right|$. Since the coefficients of (8) are real, its complex zeros occur in conjugate pairs. Hence, the one remaining zero of (8), occurring outside the contour Γ, must be real. Furthermore, it can be shown that $\xi_0 > b$ for $b > 0$ and $|\xi_0| > |\frac{ab}{a+1}|$ for $b < 0$.

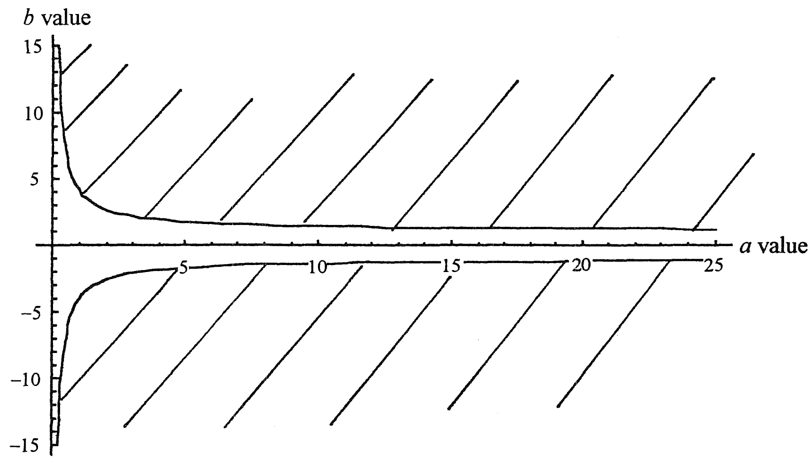

FIGURE 1. The Convergence Region (7)

Utilizing (10) and the conjectured result (12), it may be seen that these would imply

$$\sum_{r=0}^{[n/(a+1)]} \binom{n-ar}{r} b^{(n-ar)} + \sum_{r=\left[\frac{n+1}{a}\right]}^{\infty} \binom{n-ar}{r} b^{(n-ar)} = \frac{\xi_0^{n+1}}{(a+1)\xi_0 - ab},$$

so

$$\sum_{r=\left[\frac{n+1}{a}\right]}^{\infty} \binom{-(n-ar)}{r} b^{-(n-ar)} = -\sum_{j=1}^{a} \frac{\xi_j^{n+1}}{(a+1)\xi_j - ab}$$

such that

$$\sum_{r=\left[\frac{n+1}{a}\right]}^{\infty} (-1)^{r+1} \binom{ar+r-1-n}{r} b^{-(n-ar)} = -\sum_{j=1}^{a} \frac{\xi_j^{n+1}}{(a+1)\xi_j - ab},$$

where use is made of the relation (see [3])

$$\binom{-m}{n} = (-1)^n \binom{m+n-1}{n} \quad \text{and} \quad \binom{0}{n} = 0. \tag{13}$$

4. PROOF OF CONJECTURE

Consider equation (12) and let $n = -aN$ such that

$$\sum_{r=0}^{\infty} \binom{-a(N+r)}{r} b^{-a(N+r)} = \frac{\xi_0^{-aN+1}}{(a+1)\xi_0 - ab}. \tag{14}$$

Utilizing the result

$$b^{-a(N+r)} = \left(\frac{1+\xi_0^a}{\xi_0^{1+a}} \right)^{a(N+r)}$$

from (8) and equation (13) allows the left-hand side of (14) to be expressed as

$$\begin{aligned}\sum_{r=0}^{\infty}\binom{-a(N+r)}{r}b^{-a(N+r)} &= \sum_{r=0}^{\infty}(-1)^r\binom{aN+ar+r-1}{r}\left(\frac{1+\xi_0^a}{\xi_0^{1+a}}\right)^{a(N+r)} \\ &= \sum_{r=0}^{\infty}(-1)^r\binom{aN+ar+r-1}{r}\sum_{k=0}^{a(N+r)}\binom{aN+ar}{k}\xi_0^{ak-a(1+a)(N+r)}.\end{aligned} \tag{15}$$

The convergent double sum (15) may be written term by term as

$$\begin{aligned}&\binom{aN-1}{0}\left[\binom{aN}{0}\xi_0^{-a(1+a)N}+\cdots+\binom{aN}{aN-1}\xi_0^{-a(N+1)}+\binom{aN}{aN}\xi_0^{-a(N+0)}\right] \\ &-\binom{aN+a}{1}\left[\binom{aN+a}{0}\xi_0^{-a(1+a)(N+1)}+\cdots+\binom{aN+1}{aN+a-1}\xi_0^{-a(N+2)}+\binom{aN+a}{aN+a}\xi_0^{-a(N+1)}\right] \\ &+\binom{aN+2a+1}{2}\left[\binom{aN+2a}{0}\xi_0^{-a(1+a)(N+2)}+\cdots+\binom{aN+2a}{aN+2a-1}\xi_0^{-a(N+3)}+\binom{aN+2a}{aN+2a}\xi_0^{-a(N+2)}\right] \\ &-\binom{aN+3a+2}{0}\left[\binom{aN+3a}{0}\xi_0^{-a(1+a)(N+3)}+\cdots+\binom{aN+3a}{aN+3a-1}\xi_0^{-a(N+4)}+\binom{aN+3a}{aN+3a}\xi_0^{-a(N+3)}\right] \\ &+\cdots\end{aligned} \tag{16}$$

Summing (16) diagonally from the top right-hand corner and gathering the coefficient of inverse powers of ξ_0 gives

$$\sum_{r=0}^{\infty}\xi_0^{-a(N+r)}\sum_{k=0}^{r}(-1)^{r-k}\binom{a(N+r-k)}{a(N+r-k)-k}\binom{a(N+r-k)+r-k-1}{r-k}. \tag{17}$$

After some lengthy algebra, (16) may be written as

$$\begin{aligned}\xi_0^{-aN}\left[1+a\sum_{r=1}^{\infty}(-1)^r(1+a)^{r-1}\xi_0^{-ar}\right] &= \xi_0^{-aN}\left[\frac{1+\xi_0^a}{(1+a)+\xi_0^a}\right] \\ &= \xi_0^{-aN+1}\left[\frac{1}{(1+a)\xi_0-a\left(\frac{\xi_0^{1+a}}{1+\xi_0^a}\right)}\right] = \frac{\xi_0^{-aN+1}}{(1+a)\xi_0-ab},\end{aligned}$$

which is identical to the right-hand side of (14); hence, the conjecture is proved.

Some numerical results of the conjecture, to five significant digits, are shown in the following table.

n	a	b	ξ_0	Sum and Right-hand Side of (12)
3	3	e	2.83729	20.28791
3	3	$-e$	−2.55538	−20.63241
3	4	1.9	2.01521	6.66073
3	4	−1.9	−2.01521	−6.66073

5. OBSERVATIONS

1. In the special case in which $a=1$, $b=1$, the two zeros of (8) are the Golden ratio $\alpha=\xi_0=(1+\sqrt{5})/2$ and $\beta=\xi_1=(1-\sqrt{5})/2$ and equation (1) is the Fibonacci sequence. From (10), the familiar relationship
$$f_n=\sum_{r=0}^{[n/2]}\binom{n-ar}{r}=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$
is obtained.
2. Other parameter values (a, b, c) may be taken so that the solution of (1) may involve known polynomial solutions, such as the Tchebycheff polynomials.
3. In equation (12) the restriction of (n, a) being natural numbers can be relaxed to (n, a) being real numbers, in which case the combinatorial relation would involve Gamma functions.
4. For $n\geq a$, the closed-form expression at (12), namely, $\xi_0^{n+1}/[(1+a)\xi_0-ab]$ is, in fact, a solution to the difference-delay equation (1); this may be verified by direct substitution.
5. Equation (1) may be extended easily to consider a forcing term of the type $w_n=\binom{n}{m}b^n$, for example, for m and n positive integers.

6. CONCLUSIONS

A technique has been demonstrated whereby closed-form representation of infinite series may be determined. The method described in this paper may be modified and utilized to consider difference-delay equations of higher order, nonhomogeneous difference-delay equations, equations with poles of multiple order, and equations with multiple delay. These variations will be considered by the authors in a forthcoming paper. The authors [2] also considered differential difference equations in which case resulting series were able to be represented in closed form.

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