A GENERALIZATION OF STIRLING NUMBERS

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(Submitted September 1996-Final Revision December 1996)

1. INTRODUCTION

Let $W(x)$, $f(x)$, $g(x)$ be formal power series with complex coefficients, and $W(x)\neq 0$, $W(0)=1$, $f(0)=g(0)=0$, Then the coefficients $\{B_1(n,k),\ B_2(n,k)\}$ in the following expansions,

$$W(x)(f(x))^k/k!=\sum_{n\geq k}B_1(n,k)x^n/n!,\quad (g(x))^k/[W(g(x))k!]=\sum_{n\geq k}B_2(n,k)x^n/n!, \tag{1}$$

are called a weighted Stirling pair if $f(g(x))=g(f(x))=x$, i.e., f and g are reciprocal.

When $W(x)\equiv 1$, $B_1(n,k)$ and $B_2(n,k)$ reduce to a Stirling type pair whose properties are exhibited in [7].

In this paper, we shall present a weighted Stirling pair that includes some previous generalizations of Stirling numbers as particular cases. Some related combinatorial and arithmetic properties are also discussed.

2. A WEIGHTED STIRLING PAIR

Let t, α, β be given complex numbers with $\alpha\cdot\beta\neq 0$. Let $f(x)=[(1+\alpha x)^{\beta/\alpha}-1]/\beta$, $g(x)=[(1+\beta x)^{\alpha/\beta}-1]/\alpha$, and $W(x)=(1+\alpha x)^{t/\alpha}$. Then, in accordance with (1), by noting that $f(x)$ and $g(x)$ are reciprocal, we have a weighted Stirling pair, denoted by

$$\{S(n,k,\alpha,\beta;t),S(n,k,\beta,\alpha;-t)\}=\{B_1(n,k),B_2(n,k)\}.$$

We call it an $(\alpha,\beta;t)$ [resp. a $(\beta,\alpha;-t)$] pair for short. Moreover, one of the parameters α or β may be zero by considering the limit process. For instance, a $(1,0;0)$ [resp. a $(0,1;0)$] pair is just Stirling numbers of the first and second kinds.

Note that from the definition of an $(\alpha,\beta;t)$ pair and the first equation in (1), we may obtain the double generating function of $S(n,k,\alpha,\beta;t)$ as

$$(1+\alpha x)^{t/\alpha}\exp\left\{u\frac{(1+\alpha x)^{\beta/\alpha}-1}{\beta}\right\}=\sum_{n,k}S(n,k,\alpha,\beta;\,t)\frac{x^n}{n!}u^k. \tag{2}$$

If we differentiate both sides of (2) on x, then multiply by $(1+\alpha x)$ and compare the coefficients of x^nu^k, we have

$$S(n,k-1,\alpha,\beta;t+\beta)=S(n+1,k,\alpha,\beta;t)+(n\alpha-t)S(n,k,\alpha,\beta;t), \tag{3}$$

and if we differentiate both sides of (2) on u and then compare the coefficients of x^nu^k, we have

$$S(n,k,\alpha,\beta;t+\beta)=\beta(k+1)S(n,k+1,\alpha,\beta;t)+S(n,k,\alpha,\beta;t). \tag{4}$$

Thus, the recurrence relation satisfied by $S(n,k,\alpha,\beta;t)$ may be obtained by combining (3) and (4):

$$S(n+1,k,\alpha,\beta;t)=(t+\beta k-\alpha n)S(n,k,\alpha,\beta;t)+S(n,k-1,\alpha,\beta;t). \tag{5}$$

The initial values of $S(n, k, \alpha, \beta; t)$ may be verified easily from (1) because $S(n, 0, \alpha, \beta; t) = t(t-\alpha)(t-2\alpha)\cdots(t-(n-1)\alpha)$ for $n \geq 1$, $S(n, n, \alpha, \beta; t) = 1$ for $n \geq 0$, and $S(n, k, \alpha, \beta; t) = 0$ for $k > n$. Thus, a table of values of $S(n, k, \alpha, \beta; t)$ can be given by concrete computations.

TABLE 1. $\boldsymbol{S(n, k, \alpha, \beta; t)}$

$n \backslash k$	0	1	2	3
0	1			
1	t	1		
2	$t(t-\alpha)$	$2t+\beta-\alpha$	1	
3	$t(t-\alpha)$	$(t+\beta-2\alpha)+t(t-\alpha)$	$3t+3\beta-3\alpha$	1

From (2), we may get the explicit expression for $S(n, k, \alpha, \beta; t)$ via the generalized binomial theorem along the lines of (4.1) in [6].

For a complex number a, define the generalized factorial of x with increment a by $(x|a)_n = x(x-a)(x-2a)\cdots(x-na+a)$ for $n = 1, 2, \ldots$, and $(x|a)_0 = 1$.

Theorem 1: The $(\alpha, \beta; t)$ pair defined by (1) may also be defined by the following symmetric relations:

$$((x+t)|\alpha)_n = \sum_{k=0}^{n} S(n, k, \alpha, \beta; t)(x|\beta)_k; \tag{6}$$

$$(x|\beta)_n = \sum_{k=0}^{n} S(n, k, \beta, \alpha; -t)((x+t)|\alpha)_k. \tag{7}$$

Proof: The proof of the theorem may be carried out by the same argument used by Howard [6], by showing that the sequences defined by (6) and (7) satisfy the same recurrence relations and have the same initial values as that of an $(\alpha, \beta; t)$ pair. □

Examples: Let $\lambda, \theta \neq 0$ be two complex parameters. The so-called weighted degenerate Stirling numbers $(S_1(n, k, \lambda|\theta), S(n, k, \lambda|\theta))$ were first introduced and discussed by Howard [6] with definitions

$$(1-x)^{1-\lambda}\left(\frac{1-(1-x)^{\theta}}{\theta}\right)^k = k!\sum_{n\geq k} S_1(n, k, \lambda|\theta)\frac{x^n}{n!}$$

and

$$(1+\theta x)^{\mu\lambda}((1+\theta x)^{\mu}-1)^k = k!\sum_{n\geq k} S(n, k, \lambda|\theta)\frac{x^n}{n!},$$

where $\theta\mu = 1$. Now it is clear that $(-1)^{n-k}S_1(n, k, 1, \lambda|\theta) = S(n, k, 1, \theta; \theta-\lambda)$ and $S(n, k, \lambda|\theta) = S(n, k, \theta, 1; \lambda)$.

The limiting case $\theta = 0$, $\lambda \neq 0$, gives the weighted Stirling numbers $(R_1(n, k, \lambda), R_2(n, k, \lambda))$ discussed by Carlitz ([2], [3]) with definitions

$$(1-x)^{-\lambda}(-\log(1-x))^k = k!\sum_{n\geq k} R_1(n, k, \lambda)\frac{x^n}{n!}$$

and

$$e^{\lambda x}(e^x-1)^k = k!\sum_{n\ge k} R_2(n,k,\lambda)\frac{x^n}{n!},$$

where the weight function $e^{\lambda x}$ comes from the limit of $(1+\theta t)^{\lambda/\theta}$ as $\theta \to 0$. It is apparent that $((-1)^{n-k}R_1(n,k,\lambda),\ R_2(n,k,\lambda))$ forms a $(1,0;-\lambda)$ pair.

Further examples are the degenerate Stirling numbers [1] defined by

$$\left(\frac{1-(1-t)^\theta}{\theta}\right)^k = k!\sum_{n\ge k} S_1(n,k\,|\,\theta)\frac{t^n}{n!}$$

and

$$((1+\theta t)^\mu - 1)^k = k!\sum_{n\ge k} S(n,k\,|\,\theta)\frac{t^n}{n!},$$

where $\theta\mu = 1$. It is clear that $((-1)^{n-k}S_1(n,k\,|\,\theta),\ S(n,k\,|\,\theta))$ is a $(1,\theta;0)$ pair.

The noncentral Stirling numbers were first introduced by Koutras in [8] with the definitions:

$$(t)_n = \sum_{k=0}^n s_a(n,k)(t-a)^k;$$

$$(t-a)^n = \sum_{k=0}^n S_a(n,k)(t)_k.$$

It is now clear by Theorem 1 that $(s_a(n,k),\ S_a(n,k))$ is a $(1,0;a)$ pair.

3. REPRESENTATIONS OF WEIGHTED STIRLING PAIRS

For $r\ge 0$, $f_r \ne 0$, let $F(x) = \sum_{k=r}^\infty f_k x^k/k!$ and $W(x) = \sum_{j=0}^\infty W_j x^j/j!$ be two formal power series. Following Howard [6], for complex z, we define the weighted potential polynomial $F_k(z)$ by

$$W(x)\left(\frac{f_r x^r/r!}{F(x)}\right)^z = \sum_{k=0}^\infty F_k(z)x^k/k!. \tag{8}$$

Moreover, if $r\ge 1$, define the weighted exponential Bell polynomial $B_{n,k}(0,\ldots,0,f_r,f_{r+1},\ldots)$ by

$$W(x)[F(x)]^k = k!\sum_{n=0}^\infty B_{n,k}(0,\ldots,0,f_r,f_{r+1},\ldots)x^n/n!. \tag{9}$$

The following lemma is due to Howard ([6], Th. 3.1).

Lemma 2: With $F_k(z)$ and $B_{n,k}$ defined above, we have

$$\binom{k-z}{k}F_k(z) = \sum_{j=0}^k \left(\frac{r!}{f_r}\right)^j \binom{k+z}{k-j}\binom{k-z}{k+j}\frac{(k+j)!}{(k+rj)!}B_{k+rj,j}(0,\ldots,0,f_r,f_{r+1},\ldots).$$

Now, from (9) with $W(x) = (1+\alpha x)^{t/\alpha}$ and $F(x) = [(1+\alpha x)^{\beta/\alpha}-1]/\beta$, we have

$$S(n,k,\alpha,\beta;t) = B_{n,k}(1,\beta-\alpha,(\beta-\alpha)(\beta-2\alpha),(\beta-\alpha)(\beta-2\alpha)(\beta-3\alpha),\ldots). \tag{10}$$

Define the weighted potential polynomials $A_k(z)$ by

$$(1+\alpha x)^{t/\alpha}\left(\frac{\beta x}{(1+\alpha x)^{\beta/\alpha}-1}\right)^z = \sum_{k=0}^{\infty} A_k(z)\frac{x^k}{k!}, \tag{11}$$

If we differentiate both sides of (11) with respect to x, then multiply by $1+\alpha x$ and compare the coefficients of x^k, we obtain

$$zA_k(z+1) = (z-k)A_k(z) + k(t+(\alpha-\beta)z-(k-1)\alpha)A_{k-1}(z).$$

It follows that

$$\begin{aligned}(-1)^k\binom{k-n-1}{k}A_k(n+1) = (-1)^k\binom{k-n}{k}A_k(n) + (t+(\alpha-\beta)n \\ -(k-1)\alpha)(-1)^{k-1}\binom{k-n-1}{k-1}A_{k-1}(n),\end{aligned} \tag{12}$$

with initial conditions

$$\binom{-n-1}{0}A_0(n+1) = 1, \text{ for } n \geq 0, \tag{13}$$

and

$$(-1)^n\binom{-1}{n}A_n(n+1) = (t+\alpha-\beta)(t+\alpha-2\beta)\cdots(t+\alpha-n\beta), \text{ for } n \geq 1. \tag{14}$$

Therefore, by equations (12)–(14), and the recurrence relations satisfied by $S(n, n-k, \beta, \alpha; t+\alpha-\beta)$ [may be deduced from (5)] and its initial values, we have that

$$(-1)^k\binom{k-n-1}{k}A_k(n+1) = S(n, n-k, \beta, \alpha; t+\alpha-\beta).$$

It then follows from Lemma 2, by taking $r=1$ and (10) that

$$S(n, n-k, \beta, \alpha; t+\alpha-\beta) = \sum_{j=0}^{k}(-1)^j\binom{k+n+1}{k-j}\binom{k-n-1}{k+j}S(k+j, j, \alpha, \beta; t).$$

By symmetry, we have the following representation formulas for weighted Stirling pairs.

Theorem 3: For $S(n, k, \alpha, \beta; t)$ defined by (1) and $S(n, k, \beta, \alpha; t+\alpha-\beta)$ defined in a like way, we have

$$S(n, k, \alpha, \beta; t) = \sum_{j=0}^{n-k}(-1)^j\binom{2n-k+1}{n-k-j}\binom{n+j}{n-k+j}S(n-k+j, j, \beta, \alpha; t+\alpha-\beta) \tag{15}$$

and

$$S(n, k, \beta, \alpha; t+\alpha-\beta) = \sum_{j=0}^{n-k}(-1)^j\binom{2n-k+1}{n-k-j}\binom{n+j}{n-k+j}S(n-k+j, j, \alpha, \beta; t). \tag{16}$$

Remark: It should be pointed out that similar representation results for the particular case when $\alpha=\theta$, $\beta=1$, and $t=1-\lambda$ has been proved by Howard [6]. Here we borrow his proof techniques.

4. CONGRUENCE PROPERTIES OF WEIGHTED STIRLING PAIRS

A formal power series $\phi(x) = \sum_{n\geq 0} a_n x^n/n!$ is called a Hurwitz series if all of its coefficients are integers. It is well known that, for the Hurwitz series $\phi(x)$ with $a_0 = 0$, the series $(\phi(x))^k/k!$ is again a Hurwitz series for any positive integer k.

In this section we always assume $\alpha, \beta, t \in \mathbf{Z}$. Then it is clear that both $(f(x))^k/k!$ and $(g(x))^k/k!$ in (1) are Hurwitz series, so that $S(n,k,\alpha,\beta;t)$ and $S(n,k,\beta,\alpha;-t)$ are two integer sequences.

First, let $t=0$. Then we have

Theorem 4: Let p be a prime number and let k and j be integers such that $j+1<k<p$. Then the following congruence relation holds:

$$S(p+j,k,\beta,\alpha;0) \equiv 0 \pmod{p}. \tag{17}$$

Proof: Assume first that $\alpha \not\equiv 0 \pmod{p}$. For a polynomial $\phi(x)$ of degree n in x, we may express it, using Newton's interpolation formula, in the form

$$\phi(x) = \phi(\alpha_0) + \sum_{k=1}^{n} [\alpha_0 \alpha_1 \dots \alpha_k]\{x|\alpha\}_k, \tag{18}$$

where $[\alpha_0\alpha_1\dots\alpha_k]$ denotes the divided difference at the distinct points $x=\alpha_0,\alpha_1,\dots,\alpha_k,\dots$ and $\{x|\alpha\}_k = (x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{k-1})$. Moreover, we have

$$[\alpha_0\alpha_1\dots\alpha_k] = \begin{vmatrix} 1 & \alpha_0 & \cdots & \alpha_0^{k-1} & \phi(\alpha_0) \\ 1 & \alpha_1 & \cdots & \alpha_1^{k-1} & \phi(\alpha_1) \\ & & \cdots & & \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} & \phi(\alpha_k) \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & \alpha_0 & \cdots & \alpha_0^{k-1} & \alpha_0^k \\ 1 & \alpha_1 & \cdots & \alpha_1^{k-1} & \alpha_1^k \\ & & \cdots & & \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} & \alpha_k^k \end{vmatrix}. \tag{19}$$

Now take $\phi_p(x) = (x|\beta)_p$, then $\phi_p(0)=0$. We have, by (7) and (18), that

$$S(p,k,\beta,\alpha;0) = [\alpha_0\alpha_1\dots\alpha_k], \tag{20}$$

which may be expressed as a quotient of two determinants as in (19), where $\alpha_j = j\alpha$ ($j=0, 1, 2, \dots$).

Notice that the classical argument of Lagrange that applied to the proof of

$$(x-1)\cdots(x-p+1) \equiv x^{p-1}-1 \pmod{p}$$

may also be applied to prove the relation

$$\phi_p(x) = (x|\beta)_p = x(x-\beta)\cdots(x-(p-1)\beta) \equiv x^p - \beta^{p-1}x \pmod{p}, \tag{21}$$

where the congruence relation between polynomials are defined as usual (cf. [4], pp. 86-87, Th. 112). Also, using Fermat's Little Theorem, we find

$$\phi_p(j\alpha) \equiv (j\alpha)^p - \beta^{p-1}(j\alpha) \equiv \begin{cases} j\alpha \pmod{p}, & \text{if } p|\beta, \\ 0 \pmod{p}, & \text{if } p\nmid\beta, \end{cases}$$

where $j=0,1,2,\dots$. Consequently, we obtain, with $\alpha_j = j\alpha$ for $k>1$,

$$\begin{vmatrix} 1 & \alpha_0 & \cdots & \alpha_0^{k-1} & \phi_p(\alpha_0) \\ 1 & \alpha_1 & \cdots & \alpha_1^{k-1} & \phi_p(\alpha_1) \\ & & \cdots & & \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} & \phi_p(\alpha_k) \end{vmatrix} \equiv 0 \pmod{p}.$$

Moreover, the denominator is given by

$$\begin{vmatrix} \alpha & \cdots & \alpha^{k-1} & \alpha^k \\ 2\alpha & \cdots & (2\alpha)^{k-1} & (2\alpha)^k \\ & \cdots & & \\ k\alpha & \cdots & (k\alpha)^{k-1} & (k\alpha)^k \end{vmatrix} = \alpha^{k(k+1)/2} \prod_{0\le i<j\le k}(j-i) \not\equiv 0 \text{ for } k<p \pmod{p}.$$

Thus, we have that $S(p, k, \beta, \alpha; 0) \equiv 0 \pmod{p}$ for $1<k<p$.

Furthermore, let $F(x) = (x|\beta)_{p+j}$. We then have $F(x) = \sum_{k\ge1}^{p+j} S(p+j, k, \beta, \alpha; 0)(x|\alpha)_k$ and

$$\begin{aligned} F(x) &= \phi_p(x)(x-p\beta)\cdots(x-(p+j)\beta+\beta) \\ &\equiv (x^p - \beta^{p-1}x)x(x-\beta)\cdots(x-(j-1)\beta) \pmod{p} \\ &\equiv (x^p-\beta^{p-1}x)(x^j + a_1x^{j-1} + \cdots + a_{j-1}x) \pmod{p}, \end{aligned} \tag{22}$$

where $a_1, \ldots, a_{j-1} \in \mathbf{Z}$. Consequently, we have, for $1 \le i \le p+j$,

$$F(i\alpha) \equiv \begin{cases} 0 & \pmod{p}, \quad \text{if } p\nmid\beta, \\ (i\alpha)^{j+1} + a_1(i\alpha)^j + \cdots + a_{j-1}(i\alpha)^2 & \pmod{p}, \quad \text{if } p|\beta. \end{cases}$$

Since $j<k-1$, we have

$$\begin{vmatrix} 1 & \alpha_0 & \cdots & \alpha_0^{k-1} & F(\alpha_0) \\ 1 & \alpha_1 & \cdots & \alpha_1^{k-1} & F(\alpha_1) \\ & & \cdots & & \\ 1 & \alpha_k & \cdots & \alpha_k^{k-1} & F(\alpha_k) \end{vmatrix} \equiv 0 \pmod{p},$$

where the last column is a linear combination of the first k columns modulo p.

Again, the same denominator determinant is not congruent to zero modulo p for $k<p$. Thus, we have that $S(p+j, k, \beta, \alpha; 0) \equiv 0 \pmod{p}$ for $j+1<k<p$.

The case for $\alpha \equiv 0 \pmod{p}$ may be proved directly using (7), (21), and (22) by comparing the corresponding coefficients of powers of x in both sides of (21) and (22). Hence, the theorem is proved. □

Note that in the particular case in which $\alpha=1$, $\beta=0$ or $\beta=1$, $\alpha=0$, Theorem 4 reduces to congruences for Stirling numbers of the first and second kinds; see [5] for other congruences for Stirling numbers.

Corollary 5: Let α, β, t be integers. Then the $(\alpha, \beta; t)$ pair satisfies the basic congruence

$$S(p, k, \alpha, \beta; t) \equiv 0 \pmod{p}, \tag{23}$$

where p is a prime and $1<k<p$.

Proof: Let $W(x) = (1+\alpha x)^{t/\alpha} = \sum_{n\ge0} a_n x^n/n!$ with $a_n \in \mathbf{Z}, a_0 = 1$. Then it is clear from (1) that

$$\sum S(n, k, \alpha, \beta; t)x^n/n! = \left(\sum_{n\ge0} a_n x^n/n!\right)\left(\sum_{n\ge k} S(n, k, \alpha, \beta; 0)x^n/n!\right),$$

so that we have

$$S(p, k, \alpha, \beta; t) = \sum_{i=k}^{p} a_{p-i} S(i, k, \alpha, \beta; 0)\binom{p}{i}.$$

From Theorem 4 (taking $j=0$) and the fact that $\binom{p}{i}\equiv 0 \pmod p$ for $0<i<p$, it follows that $S(p,k,\alpha,\beta;t)\equiv 0 \pmod p$, and the corollary is proved. □

ACKNOWLEDGMENT

The author is very grateful to the referee for his valuable suggestions which considerably improved the presentation of this article.

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AMS Classification Numbers: 05A15, 11B73, 11A07

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