A METRIC RESULT CONCERNING THE APPROXIMATION OF REAL NUMBERS BY CONTINUED FRACTIONS

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(Submitted October 1996)

1. INTRODUCTION AND STATEMENT OF RESULTS

A basic result in the theory of Diophantine approximations of irrational numbers by rationals, where certain additional congruence relations are satisfied, is given by the following theorem of Uchiyama [8]:

For any irrational number ξ, any $s > 1$, and integers a and b, there are infinitely many integers u and $v \neq 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| < \frac{s^2}{4v^2} \tag{1.1}$$

and

$$u \equiv a \bmod s, \quad v \equiv b \bmod s \tag{1.2}$$

provided that a and b are not both divisible by s.

Theorems of the same type with greater constants on the right-hand side of (1.1) were proved by Hartman [3] and Koksma [6]. Recently, the author has shown [1] that the theorem of Uchiyama no longer holds if the constant $s^2/4$ is replaced by any smaller number. Assuming weaker arithmetical restrictions in (1.2) on numerators and denominators of the approximants, the constant in (1.1) can be diminished. For prime moduli p, the author has shown (see [2]):

Let $0 < \varepsilon \leq 1$, and let p be a prime with $p > (2/\varepsilon)^2$; h denotes any integer that is not divisible by p. Then, for any real irrational number ξ, there are infinitely many integers u and $v > 0$ satisfying

$$\left|\xi - \frac{u}{v}\right| \leq \frac{(1+\varepsilon)p^{3/2}}{\sqrt{5}\,v^2} \tag{1.3}$$

and

$$u \equiv hv \not\equiv 0 \bmod p. \tag{1.4}$$

The object of this paper is to show that almost all irrational numbers (in the sense of the Lebesgue-measure) are better approximated by fractions u/v satisfying (1.4).

Theorem 1.1: Let $\varepsilon > 0$, and let p be any prime. Then there is a set $\mathscr{A} \subset (0, 1)$ of measure 1 depending at most on ε and p, such that every real number ξ from $\mathscr{A}$ satisfies the following conditions:

If h denotes any integer that is not divisible by p, there are infinitely many integers u and $v > 0$ with

$$\left|\xi - \frac{u}{v}\right| < \frac{\varepsilon}{v^2} \tag{1.5}$$

and

$$u \equiv hv \not\equiv 0 \mod p. \tag{1.6}$$

The main tool used in proving this theorem is a certain generalization of the famous average-theorem of Gauss-Kusmin-Lévy concerning the elements of continued fractions (see Satz 35 in [4]), which is stated in Lemma 2.1 below. It follows from [5] or [7].

The set $\mathscr{A}$ given in Theorem 1.1 depends on ε and p. One may ask whether there are irrationals, where (1.5) and (1.6) hold for *every* $\varepsilon > 0$ and for *every* prime p.

Theorem 1.2: There is an uncountable subset $\mathscr{B}$ of $(0, 1)$ such that, for every real number ξ from $\mathscr{B}$, for every $\varepsilon > 0$, every prime p and any integer h that is not divisible by p, the inequality (1.5) and the congruences (1.6) hold for infinitely many integers u and $v > 0$.

2. PROOF OF THE THEOREMS

Lemma 2.1: Let $r_1, r_2, \ldots, r_k$ $(k \ge 1)$ be positive integers. Then the successive elements $r_1, r_2, \ldots, r_k$ occur infinitely often in the sequence $a_1, a_2, a_3, \ldots$ of the continued fraction expansion $\langle 0; a_1, a_2, a_3, \ldots \rangle$ of almost all real numbers from $(0, 1)$.

Proof of Theorem 1.1: Let $\varepsilon > 0$ and p be any prime. Moreover, let $s > 0$ be some integer with

$$\frac{1}{sp} < \varepsilon. \tag{2.1}$$

By Lemma 2.1, there is a subset $\mathscr{A} \subset (0, 1)$ of measure 1 such that the finite sequence

$$\underbrace{sp+1, sp, sp+1, sp, \ldots, sp+1, sp}_{2p} \tag{2.2}$$

occurs infinitely often among the elements of the continued fraction expansion of every number from $\mathscr{A}$. Obviously, $\mathscr{A}$ depends on ε and p.

Now we fix some number ξ from $\mathscr{A}$; by $\xi = \langle 0; a_1, a_2, a_3, \ldots \rangle$ we denote its continued fraction expansion. There are infinitely many integers $j \ge 1$ such that, for all integers n with $1 \le n \le 2p$, we have

$$a_{j+n+2} = \begin{cases} sp+1, & \text{if } n \equiv 1 \mod 2, \\ sp, & \text{if } n \equiv 0 \mod 2. \end{cases} \tag{2.3}$$

Let j be such an index, and put

$$w_n := p_{j+n} - hq_{j+n}, \tag{2.4}$$

where $h \not\equiv 0 \mod p$ denotes some integer. The integers p_m and q_m are given by

$$\begin{aligned} p_{-1} &:= 1, & p_0 &:= 0, & p_{m+2} &:= a_{m+2}p_{m+1} + p_m \quad (m \ge -1), \\ q_{-1} &:= 0, & q_0 &:= 1, & q_{m+2} &:= a_{m+2}q_{m+1} + q_m \quad (m \ge -1). \end{aligned} \tag{2.5}$$

It follows easily from (2.5) that w_n, w_{n+1}, and w_{n+2} satisfy

$$w_{n+2} = a_{j+n+2}w_{n+1} + w_n \quad (1 \le n \le 2p). \tag{2.6}$$

We shall show by mathematical induction that the following congruences hold for all odd integers n with $1 \le n \le 2p-1$:

$$w_{n+1} \equiv w_2 \bmod p \quad \text{and} \quad w_{n+2} \equiv w_1 + \frac{n+1}{2}w_2 \bmod p. \tag{2.7}$$

For $n = 1$ we have, by (2.3) and (2.6),

$$w_3 = a_{j+3}w_2 + w_1 \equiv w_1 + \frac{1+1}{2}w_2 \bmod p.$$

Now assume that (2.7) holds for some odd integer n with $1 \le n \le 2p-3$. Since $n+3$ is some even integer and since $3 < n+3 < 2p+2$, we conclude from (2.3), (2.6), and the induction hypothesis that

$$w_{n+3} = a_{j+n+3}w_{n+2} + w_{n+1} \equiv w_{n+1} \equiv w_2 \bmod p. \tag{2.8}$$

Similarly, since $n+4$ is some odd integer with $3 < n+4 < 2p+2$,

$$\begin{aligned} w_{n+4} &= a_{j+n+4}w_{n+3} + w_{n+2} \equiv w_{n+3} + w_{n+2} \\ &\overset{(2.7),(2.8)}{\equiv} w_2 + \left(w_1 + \frac{n+1}{2}w_2\right) = w_1 + \frac{n+3}{2}w_2 \bmod p. \end{aligned}$$

Hence, (2.7) holds for all odd integers $1 \le n \le 2p-1$.

In what follows, we distinguish two cases:

Case 1. $w_2 \equiv 0 \bmod p$.

This means, by (2.4),

$$p_{j+2} \equiv hq_{j+2} \bmod p. \tag{2.9}$$

Case 2. $w_2 \not\equiv 0 \bmod p$.

Since p is a prime, we even have $(p, w_2) = 1$. We write down the right-hand congruences in (2.7) for all odd integers $1 \le n \le 2p-1$:

$$\left.\begin{array}{rcl} w_3 & \equiv & w_1 + w_2 \\ w_5 & \equiv & w_1 + 2w_2 \\ w_7 & \equiv & w_1 + 3w_2 \\ w_9 & \equiv & w_1 + 4w_2 \\ & \vdots & \\ w_{2p+1} & \equiv & w_1 + pw_2 \end{array}\right\} \bmod p.$$

By $(p, w_2) = 1$, the p integers $w_3, w_5, w_7, \ldots, w_{2p+1}$ represent p distinct residue classes with respect to the modulus p. Hence, there is some odd integer k with $3 \le k \le 2p+1$ and $w_k \equiv 0 \bmod p$, which means

$$p_{j+k} \equiv hq_{j+k} \bmod p. \tag{2.10}$$

Collecting together from (2.9) and (2.10), we have proved the existence of an integer k with $2 \le k \le 2p+1$ and

$$p_{j+k} \equiv hq_{j+k} \not\equiv 0 \bmod p. \tag{2.11}$$

Since h is not divisible by the prime p, the assumption $p_{j+k} \equiv hq_{j+k} \equiv 0 \mod p$ would imply that $(p_{j+k}, q_{j+k}) > 1$, which is well known to be impossible. Finally we get, by (2.1), (2.3), and (2.5),

$$\left| \xi - \frac{p_{j+k}}{q_{j+k}} \right| < \frac{1}{q_{j+k}q_{j+k+1}} \le \frac{1}{a_{j+k+1}q_{j+k}^2} \le \frac{1}{spq_{j+k}^2} < \frac{\varepsilon}{q_{j+k}^2}; \tag{2.12}$$

note that $3 \le k+1 \le 2p+2$. The integers j from (2.3) are not bounded, hence, by (2.11) and (2.12), there are infinitely many integers $u := p_{j+k}$ and $v := q_{j+k} > 0$ satisfying the conditions (1.5) and (1.6) of the theorem. Thus, the proof is complete.

Examples:

1.) Let $\varepsilon = 1$; (2.1) holds for every prime p with $s = 1$. Hence, the number

$$\xi := \langle 0; \overline{1, p} \rangle = \frac{\sqrt{p^2+4p} - p}{2}$$

belongs to the corresponding set $\mathcal{A}$ in Theorem 1.1.

2.) It is well known that

$$\left| \frac{1+\sqrt{5}}{2} - \frac{F_{n+1}}{F_n} \right| < \frac{1}{F_n F_{n+1}} \quad (n \ge 1).$$

Since F_n and F_{n+1} are coprime integers, and because any prime p divides infinitely many Fibonacci numbers $F_{n-1} = F_{n+1} - F_n$, for every prime p the congruence $F_n \equiv F_{n+1} \not\equiv 0 \mod p$ is satisfied for infinitely many pairs of Fibonacci numbers F_n and F_{n+1}.

Proof of Theorem 1.2: Let $\mathcal{B} \subset (0, 1)$ be the subset of those numbers $\xi = \langle 0; a_1, a_2, a_3, \ldots \rangle$, where the elements $a_1, a_2, a_3, \ldots$ of the continued fraction expansions are given by the following scheme:

$$\begin{array}{ccccccccc} a_2 & a_4 & & & & & & & \\ a_6 & a_8 & a_{10} & a_{12} & a_{14} & & & & \\ a_{16} & a_{18} & a_{20} & a_{22} & a_{24} & a_{26} & a_{28} & a_{30} & a_{32} \\ & & & & \vdots & & & & \end{array} =$$

$$\begin{array}{llllllllllllll} \alpha_1 \cdot 1 \cdot 2 & 1 \cdot 2 & & & & & & & & & & & & \\ \alpha_2 \cdot 2 \cdot 2 & 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 3 & 2 \cdot 3 & & & & & & & & & \\ \alpha_3 \cdot 3 \cdot 2 & 3 \cdot 2 & 3 \cdot 3 & 3 \cdot 3 & 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 4 & 3 \cdot 4 & 3 \cdot 4 & & & & & \\ \alpha_4 \cdot 4 \cdot 2 & 4 \cdot 2 & 4 \cdot 3 & 4 \cdot 3 & 4 \cdot 3 & 4 \cdot 4 & 4 \cdot 4 & 4 \cdot 4 & 4 \cdot 4 & 4 \cdot 5 & 4 \cdot 5 & 4 \cdot 5 & 4 \cdot 5 & 4 \cdot 5 \\ \alpha_5 \cdot 5 \cdot 2 & 5 \cdot 2 & \cdots & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & \end{array}$$

In the first row, the integers $\alpha_1, \alpha_2, \alpha_3, \ldots$ are arbitrary numbers from the set $\{1, 2\}$. Let the elements a_m with odd indices m be given by

$$a_m = 1 + a_{m+1} \quad (m \equiv 1 \mod 2,\ m \ge 1).$$

Obviously, $\mathcal{B}$ is an uncountable set of numbers, since the set of sequences with elements from $\{1, 2\}$ is not countable.

If ξ belongs to $\mathcal{B}$ and if p is some prime, there are infinitely many finite sequences $r_1, r_2, r_3, \ldots, r_{2p}$ among the elements of the continued fraction expansion of ξ, where

$$r_\nu \equiv 1 \mod p \quad (\nu \equiv 1 \bmod 2),$$
and
$$r_\nu \equiv 0 \mod p \quad (\nu \equiv 0 \bmod 2).$$

Moreover, we can choose all the integers $r_1, r_2, r_3, \ldots, r_{2p}$ as large as possible for infinitely many such finite sequences.

Now infinitely many integers u and $v > 0$ satisfying (1.5) and (1.6) can be found in the same way as was shown in the proof of Theorem 1.1.

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AMS Classification Numbers: 11J04, 11J70, 11J83

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