A GENERALIZATION OF THE KUMMER IDENTITY AND ITS APPLICATION TO FIBONACCI-LUCAS SEQUENCES

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1. INTRODUCTION

We assume that the reader is familiar with the basic notations and facts from combinatorial analysis (cf. [1]). In [2] and [4], H. W. Gould and P. Haukkanen discussed the following transformation of the sequence $\{A_k\}_{k=0}^{\infty}$, that is,

$$S(n)=\sum_{k=0}^{n}\binom{n}{k}t^{n-k}s^k A_k \quad \text{for } n=0,1,2,\dots. \tag{1}$$

Let $A(x)$ and $S(x)$ be the formal series determined by $\{A_k\}_{k=0}^{\infty}$ and $\{S(k)\}_{k=0}^{\infty}$. Then

$$S(x)=\frac{1}{1-tx}A\left(\frac{sx}{1-tx}\right). \tag{2}$$

They found that transformation of (1) or (2) is related to Fibonacci numbers. Recently, Shapiro et al. [8] and Sprugnoli [10] introduced the theories of the Riordan array and the Riordan group, respectively, in an effort to answer the following question: What are the conditions under which a combinatorial sum can be evaluated by transforming the generating function? We think that the works of Gould and Haukkanen mentioned above can be extended by using the Riordan group or the Riordan array. We adopt the concept of the Riordan group in this paper because both theories are essentially the same. It is certain that the idea of the Riordan group can be traced back to Mullin and Rota [5], Rota [6], and Roman and Rota [7]. The reader is referred to [5]-[10] for more details. In the present paper we are concerned with the following identity, called the *Kummer identity*:

$$x^n+y^n=\sum_{0\le k\le n/2}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(x+y)^{n-2k}(xy)^k. \tag{3}$$

It is well known that the number $(-1)^k\frac{n}{n-k}\binom{n-k}{k}$ is closely related to the *probléme des ménages* and the Kummer identity is closely related to the Fibonacci-Lucas sequence (cf. [11]). With the help of the Riordan group, we give a generalization of the Kummer identity as follows.

Theorem 1: $\forall\, x, y$ and $z\in R$ with $xy+yz+zx=0$, the following identity holds:

$$x^n+y^n+z^n=\sum_{0\le k\le n/3}\frac{n}{n-2k}\binom{n-2k}{k}(x+y+z)^{n-3k}(xyz)^k. \tag{4}$$

It is reasonable to believe that the above identity can find some application to the second-order Fibonacci-Lucas sequence as the Kummer identity does; this needs to be discussed further.

Remark: According to the referee, it is worth noting that the *Kummer identity* and Theorem 1 can also be obtained by the use of symmetric functions, a large and older literature that dates back to Albert Girard in (1629) and concerns summation formulas for the powers of the roots of algebraic equations. Actually, in view of symmetric functions, these results may be extended in various forms to deal with $t_1^n + t_2^n + \cdots + t_n^n$, where t_i are the distinct roots of certain algebraic equations. However, not completely new as it is, we rederive it by using the Riordan group for the purpose of establishing some combinatorial sums as well as a reciprocal relation satisfied by two famous numbers in the literature.

To make this paper self-contained, we need some elementary results regarding the Riordan group.

Definition 1: Given $d(t),\ f(t) \in R[t]$ with $f(0) = 0$. Let $d_{n,k}$ be the number given by

$$d_n = [t^n](d(t)f^k(t)), \tag{5}$$

where $[t^n](\cdot)$ denotes the coefficient of t^n in $(\cdot)$. Write $M = (d(t), f(t))$ for matrix $(d_{n,k})$ with entries in R, and M_R for the set $\{M = (d(t), f(t)) \mid f(0) = 0\}$. Then M_R is referred to as the Riordan group endowed with the binary operation $*$ as follows:

$$(d(t), f(t)) * (h(t), g(t)) = (d(t)h(f(t)), g(f(t))), \tag{6}$$

where $h(f(t))$ denotes the composition of $h(t)$ and $f(t)$ just as usual.

Theorem 2: (Cf. [9].) Let $d_{n,k}$ be defined by (5) and $F(t) = \sum_{k\ge 0} a_k t^k$. Then

$$\sum_{n\ge k\ge 0} d_{n,k} a_k = [t^n]d(t)F(f(t)). \tag{7}$$

As far as the Kummer identity (3) and its generalization (4) are concerned, we know from [9] that

$$(d_{n,k}) = \left(\frac{k}{n-k}\binom{n-k}{k}\right) \quad \text{and} \quad \left(\frac{k}{n-2k}\binom{n-2k}{k}\right)$$

are just two elements of M_R. Based on this result, (3) can be verified directly and (4) can be rediscovered.

2. PROOF OF THEOREM 1

Recall the facts mentioned in the last section. We can obtain two preliminary results directly from Theorem 2.

Lemma 1: Let $F(t) = \sum_{k\ge 1} \frac{f_k}{k} t^k$ and $G(t) = \sum_{k\ge 1} \frac{g_k}{k} t^k$. Then

$$\sum_{0\le k\le n/2} \frac{n}{n-k}\binom{n-k}{k} f_k = f_0 + n[t^n]F\left(\frac{t^2}{1-t}\right); \tag{8}$$

$$\sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k} g_k = g_0 + n[t^n]G\left(\frac{t^3}{1-t}\right). \tag{9}$$

Proof: Since

$$\binom{n-k}{k} = \frac{n-k}{k}\binom{n-k-1}{k-1}, \quad \binom{n-2k}{k} = \frac{n-2k}{k}\binom{n-2k-1}{k-1},$$

and

$$\binom{n-k-1}{k-1}=[t^n]\left(\frac{t^2}{1-t}\right)^k, \quad \binom{n-2k-1}{k-1}=[t^n]\left(\frac{t^3}{1-t}\right)^k.$$

From Theorem 2, (8) and (9) follow as desired. □

Lemma 2: $\forall\, a, b$ and $c \in R$. Then the following identities hold:

$$a^n+b^n=\sum_{0\le k\le n/2}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(ab)^k \tag{10}$$

for all a and b satisfying $a+b=1$;

$$a^n+b^n+c^n=\sum_{0\le k\le n/3}\frac{n}{n-2k}\binom{n-2k}{k}(abc)^k \tag{11}$$

for all $a+b+c=1$ and $ab+bc+ca=0$.

Proof: It suffices to show (10) because a similar argument remains valid for (11). From Lemma 1, it follows that

$$F(t)=\sum_{k\ge1}\frac{(-ab)^k}{k}t^k=-\ln(1+abt) \tag{12}$$

and

$$F\left(\frac{t^2}{1-t}\right)=\ln(1-t)-\ln(1-t+abt^2). \tag{13}$$

Consider that $a+b=1$ if and only if $1-t+abt^2=(1-at)(1-bt)$. Thus, it is easy to verify that

$$F\left(\frac{t^2}{1-t}\right)=\ln(1-t)-\ln(1-at)-\ln(1-bt) \tag{14}$$

and

$$n[t^n]f\left(\frac{t^2}{1-t}\right)=-1+a^n+b^n. \tag{15}$$

Combining the above result with (10) gives the complete proof of the Lemma. □

Proof of Theorem 1 via the Riordan Group: Let

$$a=\frac{x}{x+y+z}, \quad b=\frac{y}{x+y+z}, \quad \text{and} \quad c=\frac{z}{x+y+z}.$$

We see that $ab+bc+ca=0$ equals $xy+yz+zx=0$. Then Theorem 1 follows as desired. □

A similar proof of the Kummer identity (3) was also found by Sprugnoli in [9].

3. APPLICATION

In this section, by setting specified values a and b into Theorem 1, we will find various combinatorial identities.

Example 3.1: Let $a=(x+\sqrt{x^2-4})/2x$ and $b=(x-\sqrt{x^2-4})/2x$. Then (10) gives a generalization of the very old Hardy identity,

$$\sum_{0\le k\le n/2}(-1)^k \frac{n}{n-k}\binom{n-k}{k}x^{n-2k} = \frac{(x+\sqrt{x^2-4})^n+(x-\sqrt{x^2-4})^n}{2^n}. \tag{16}$$

Example 3.2: Let $a=t$ and $b=1-t$. Then

$$t^{2n}+(1-t)^{2n} = \sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(t(1-t))^k. \tag{17}$$

This implies that

$$\sum_{k=0}^{n}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{k}{m-k}=\binom{2n}{m}, \tag{18}$$

where m is an integer, $0\le m\le n-1$. Thus,

$$\sum_{k=0}^{n}(-1)^k \frac{\binom{2n+1}{2k+1}}{\binom{2n-1}{k}} \equiv 2. \tag{19}$$

It is worth noting that the above two identities are missing from [3].

Proof: Equation (18) can be obtained by comparing the coefficient of t^m and (19) can be obtained by integrating from 0 to 1 on both sides of (17). □

Note that the same method exhibited above is often used when we proceed to set up a combinatorial identity.

Example 3.3: Let F_n be the n^{th} Fibonacci number defined by

$$\begin{cases} F_{n+2}=F_{n+1}+F_n & (n\ge 0); \\ F_1=F_0=1. \end{cases} \tag{20}$$

Then

$$F_n+F_{n-2}=\sum_{k=0}^{n/2}\frac{n}{n-k}\binom{n-k}{k} \quad (n\ge 2). \tag{21}$$

Furthermore, we have an arithmetic identity,

$$F_p+F_{p-2}=1 \ (\operatorname{mod} p), \tag{22}$$

where p is a prime.

Now let us focus attention on (4). Without loss of generality, we suppose that x, y, and z are all roots of a given equation,

$$X^3-(c-b_1)X^2-(cb_1-b_2)X-cb_2=(X-c)(X^2+b_1X+b_2).$$

Then $xy+yz+zx=0$ is equivalent to condition $cb_1=b_2$. Thus, from the Kummer identity (3) and its generalization (4), it follows that

$$x^n+y^n=\sum_{0\le k\le n/2}(-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k}b_1^{n-2k}b_2^k \tag{23}$$

and

$$x^n+y^n+c^n=\sum_{0\le k\le n/3}\frac{n}{n-2k}\binom{n-2k}{k}(c-b_1)^{n-3k}(cb_2)^k. \tag{24}$$

Under the condition that $cb_1 = b_2$, the above identities lead to the following theorem.

Theorem 3: $\forall t_1, t_2 \in R$. Then we have

$$t_1^n + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k} t_1^k t_2^{n-k} = \sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(t_1-t_2)^{n-3k}(t_1^2 t_2)^k. \tag{25}$$

We are convinced that this result includes a series of combinatorial identities. To justify this claim, we write down some interesting identities below.

Example 3.4: Let $t_1 = 1$, $t_2 = t$. Then

$$1 + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k} t^{n-k} = \sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(1-t)^{n-3k} t^k, \tag{26}$$

which implies that

$$1 + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{(n-k)(n-k+1)}\binom{n-k}{k} = \sum_{0\le k\le n/3} \frac{n}{(n-2k)(n-2k+1)} \tag{27}$$

and

$$\sum_{0\le k\le n/3} (-1)^k \frac{n}{n-2k}\binom{n-2k}{k}\binom{n-3k}{m-k} = \frac{n}{m}\binom{m}{n-m}. \tag{28}$$

Example 3.5:

$$t^n + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{n-k}\binom{n-k}{k} t^k = \sum_{0\le k\le n/3} \frac{n}{n-2k}\binom{n-2k}{k}(t-1)^{n-3k} t^{2k}, \tag{29}$$

which implies that

$$\frac{1}{n+1} + \sum_{0\le k\le n/2} (-1)^{n-k} \frac{n}{(n-k)(k+1)}\binom{n-k}{k} = \sum_{0\le k\le n/3} (-1)^{n-3k} \frac{n\binom{n}{k}}{(n-2k)(n-k+1)\binom{n}{2k}}, \tag{30}$$

and

$$\sum_{0\le k\le n/3} (-1)^{n-k-m} \frac{n}{n-2k}\binom{n-2k}{m-k}\binom{n-3k}{m-2k} = (-1)^{n-m}\frac{n}{n-m}\binom{n-m}{m}. \tag{31}$$

Exploring further, let $t_1 = 1/\sqrt{1-x}$, $t_2 = \sqrt{1-x}$, and n be replaced by $2n$. Then Theorem 3 suggests the next example.

Example 3.6: $\forall\, 0 \le x \le 1$, the following holds:

$$\begin{aligned}(1-x)^{-n} &+ \sum_{0\le k\le n} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(1-x)^{n-k}\\ &= \sum_{0\le k\le 2n/3} \frac{n}{n-k}\binom{2n-2k}{k} x^{2n-3k}(1-x)^{-n+k}.\end{aligned} \tag{32}$$

Using the generating function technique to expand $(1-x)^{-n}$ into a formal series of the power of x, this identity gives

$$\begin{aligned}&\sum_{0\le i\le+\infty}\binom{n-1+i}{n-1}x^1+\sum_{0\le k,\,i\le n}(-1)^{k+i}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{n-k}{i}x^i\\&=\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{n-k-1+i}{n-k-1}x^{2n-3k+i}.\end{aligned}\tag{33}$$

By the same argument as taken in Example 3.2, we can derive

$$\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-2k}{k}\binom{m+2k-n-1}{n-k-1}=\binom{n-1+m}{n-1}\tag{34}$$

for all $m\ge n+1$, and

$$\begin{aligned}&\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-2k}{k}\binom{m+2k-n-1}{n-k-1}\\&=\binom{n-1+m}{n-1}+\sum_{0\le k\le n}(-1)^{m+k}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{n-k}{m}\end{aligned}\tag{35}$$

for all $m\le n$.

As mentioned earlier, the coefficient appearing on the right-hand side of the Kummer identity (3) is closely related to the number $\mu(n)$ of the reduced *ménages* problem with n married couples, for which there exists an explicit computing formula as below:

$$\mu(n)=\sum_{0\le k\le n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

In the meantime, it is also well known that, for the set $[n]$, the number $d(n)$ of derangements of $[n]$ is equal to

$$d(n)=\sum_{k=0}^{n}(-1)^k\frac{n!}{k!}.$$

The most important case related to applications of Theorem 1 is that we find the following reciprocal relation connecting $\mu(n)$ with $d(n)$ be means of (32). Indeed, the following theorem follows from (32).

Theorem 4—The Reciprocal Relation: Given $\mu(n)$ and $d(n)$ as above, we have

$$\mu(n)=\sum_{k=0}^{n}A(n,k)\,d(k)\tag{36}$$

if and only if

$$d(n)=\sum_{k=0}^{n}B(n,k)\,\mu(k),\tag{37}$$

where $A(n,k)$ and $B(n,k)$ are given, respectively, by

$$A(n,k)=\sum_{i=0}^{m}(-1)^i\frac{2n}{2n-i}\binom{2n-i}{i}\binom{n-i}{k}$$

and

$$B(n,k)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{2i}{k-i}.$$

Proof: To prove (36), we first note that

$$\sum_{0\le i\le n}\binom{n-1+i}{n-1}x^i+\sum_{0\le k,i\le n}(-1)^{k+i}\frac{2n}{2n-k}\binom{2n-k}{k}\binom{n-k}{i}x^i$$
$$=\sum_{n+i\le 3k\le 2n}\frac{n}{n-k}\binom{2n-2k}{k}\binom{n-k-1+i}{n-k-1}x^{2n-3k+i}.$$

Multiply both sides of this identity by e^x and then integrate from $-\infty$ to 1. Consider that

$$\int_{-\infty}^{1}x^ie^x\,dx=(-1)^id(i)e \quad\text{and}\quad \int_{-\infty}^{1}(1-x)^{n-k}e^x\,dx=(n-k)!\,e.$$

Set these into the above identity, to obtain

$$\mu(n)+\sum_{0\le i\le n}\binom{n-1+i}{n-1}(-1)^id(i)$$
$$=\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{n-k-1+i}{n-k-1}(-1)^{2n-3k+i}d(2n-3+i),$$

which leads to

$$\mu(n)=-\sum_{0\le i\le n}\binom{n-1+i}{n-1}(-1)^id(i)$$
$$+\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{n-k-1+i}{n-k-1}(-1)^{2n-3k+i}d(2n-3k+i)$$
$$=\sum_{0\le i\le n}\left\{\sum_{0\le k\le 2n/3}\frac{n}{n-k}\binom{2n-k}{k}\binom{i+2k-n-1}{n-k-1}-\sum_{0\le i\le n}\binom{n-1-i}{n-1}\right\}(-1)^id(i).$$

Combining the above result with (35) yields the complete proof of the theorem.

Now we proceed to prove (37). Let ϕ_1 and ϕ_2 be the two roots of the equation $X^2-xX+1=0$. For simplicity, we write H_n for $\phi_1^{2n}+\phi_2^{2n}$. Then, from the Hardy identity (16), we have

$$H_n=\sum_{0\le k\le n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}x^{2(n-k)}.$$

On the other hand, observe that $\phi_1+\phi_2=x$ and $\phi_1\phi_2=1$. We can show by induction on n that

$$x^{2n}=\sum_{0\le k\le n}\binom{2n}{n+k}H_k.$$

Thus, relation (37) reads out from the above identities immediately as desired. □

Evidently, $B(n,n)=1$, $B(n,k)=0$ $(n<k)$, which gives a new efficient recurrence relation for $\mu(n)$ as follows:

$$\mu(n)=d(n)-\sum_{0\le k\le n-1}B(n,k)\mu(k). \tag{38}$$

Besides its application to combinatorial identities, we also find that identity (4) is closely related to the second-order Fibonacci-Lucas sequence. Let $\Omega(\lambda_1, \lambda_2)$ denote the set of second-order Fibonacci-Lucas sequences, defined as follows:

$$\begin{cases} L_{n+2} = \lambda_1 L_{n+1} + \lambda_2 L_n \quad (n \geq 0); \\ L_0 = c_0, \; L_1 = c_1. \end{cases}$$

Let $x_1, x_2, x_1 \neq x_2$ be the two roots of the equation $x^2 - \lambda_1 x - \lambda_2 = 0$. Then we have

Lemma 3—De Moivre Formula: (Cf. [11].) Let $L_n^{(i-1)} \in \Omega(\lambda_1, \lambda_2)$ and x_i $(i = 1, 2)$ be given as above. Then

$$x_i^n = L_n^{(1)} x_i + L_n^{(0)}, \tag{39}$$

where $L_n^{(0)}$ and $L_n^{(1)}$ are the second-order Fibonacci-Lucas sequences with $c_0 = 1$, $c_1 = 0$, and $c_0 = 0$, $c_1 = 1$, respectively.

Let $c = t \in R$. Then the conditions that $a+b+c=1$ and $ab+bc+ca=0$ amount to the following relations:

$$\begin{cases} a+b = 1-t; \\ ab = t(t-1). \end{cases}$$

This means that a and b satisfy the equation $X^2 - (1-t)X + t(t-1) = 0$. Therefore, by means of (39), we restate identity (11) in the following form in terms of the second-order Fibonacci-Lucas sequence.

Theorem 5: Let $L_n^{(0)}(t), L_n^{(1)}(t)$ be the second-order Fibonacci-Lucas sequences with $\lambda_1 = 1-t$, $\lambda_2 = t(1-t)$. Then

$$t^n + (1-t)L_n^{(1)}(t) + 2L_n^{(0)}(t) = \sum_{0 \leq k \leq n/3} \frac{n}{n-2k} \binom{n-2k}{k} (t^2(t-1))^k. \tag{40}$$

We believe that the above result can be useful in finding some arithmetic identities for the second-order Fibonacci-Lucas sequence. For instance, we can obtain

Corollary 5.1:

$$(1-t)L_p^{(1)}(t) + 2L_p^{(0)}(t) = 1-t \pmod{p}, \tag{41}$$

where $L_p^{(0)}(t)$, $L_p^{(1)}(t)$ are as given by (39) and p is a prime.

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