MEAN CROWDS AND PYTHAGOREAN TRIPLES

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1. INTRODUCTION

The opportunity to temporarily avoid computational distractions and thereby concentrate on new concepts is helpful to many students who find themselves struggling in pre-calculus mathematics courses. For this reason, instructors in these classes often present introductory examples that have been carefully designed to require no irrational numbers.

For example, it is likely that anyone who has taught trigonometry appreciates the utility of Pythagorean triples, i.e., nontrivial integer solutions $(x, y; z)$ of $x^2+y^2=z^2$. A well-known characterization theorem for these triples [7, p. 190] enables one to generate an endless supply of nonsimilar right triangles whose angles have rational trigonometric function values.

In the spirit of Pythagorean triples, I present an analogous characterization of those pairs of positive integers whose arithmetic, geometric, harmonic, and certain other means are also integers. In this development, Pythagorean triples will also surprisingly appear in ways that go beyond this motivation by analogy.

2. PYTHAGOREAN MEANS

In his *Commentary on Euclid*, Proclus of Alexandria (ca. 410-485 A.D.) reports that, while traveling in Mesopotamia, Pythagoras learned about the theory of proportionals (means); specifically, the binary operations we call the arithmetic, geometric, and harmonic means. Later, his followers extended this concept to include several other means. The neo-Pythagorean philosopher, Nicomachus of Gerasa (ca. 100 A.D.), and the geometer, Pappus of Alexandria (ca. 300 A.D.), each presented distinct but intersecting lists of ten "means" of the Pythagoreans. Most of these probably would not qualify as means by modern standards, e.g., [11, p. 84], since they fail the (perfectly reasonable) condition that a mean should always return a value between those of its arguments. In this article I will restrict attention to those means of the Pythagoreans that satisfy this intermediacy condition [12], [13].

The first three means common to the lists of Nicomachus and Pappus are the familiar *arithmetic*, *geometric*, and *harmonic means*, defined by

$$a=A(p,y)=\frac{p+y}{2},\quad g=G(p,y)=\sqrt{py},\quad h=H(p,y)=\frac{2py}{p+y}.$$

Another mean given by both Nicomachus and Pappus is occasionally called the "subcontrary" mean, but that name has also been used for the harmonic mean, so I propose the name *tetra* mean, indicating its fourth position in both lists. It is defined by

$$t=T(p,y)=\frac{p^2+y^2}{p+y}.$$

Results relating the arithmetic, geometric, and harmonic means are abundant (e.g., [1], [3], [4], [5], [6], [10], [11], [13]. [14]). Among the best-known examples, the A-G-H Mean Inequality states that, for any p and y, $h \le g \le a$, with equality holding if and only if $p=y$. An easy exercise will extend this to $h \le g \le a \le t$. The Musical Proportionality and Golden Proportionality state that

$$\frac{y}{a} = \frac{h}{p} \quad \text{and} \quad \frac{a}{g} = \frac{g}{h},$$

respectively. The latter is the basis for the Babylonian method of extraction of square roots [2, p. 56].

To avoid unnecessary computational distractions when introducing these means, we might wish to generate nontrivial examples of positive integers whose arithmetic, geometric, and harmonic means are also integers. (A near-miss is the example $p = 6$ and $y = 12$, which gives $a = 9$, $h = 8$, and $g = 6\sqrt{2}$. The early Pythagorean number-mystic Philolaus of Tarantum saw this 6-8-12 relationship as an indication that the cube, with its 6 faces, 8 vertices, and 12 edges, was particularly harmonious [8, pp. 85-86].)

3. MEAN CROWDS

In the spirit of Pythagorean triples, let us define a *mean crowd* to be an ordered sextuple (p, y, t, h, a, g) of distinct positive integers such that $p < y$ and t, h, a, and g are, respectively, the tetra, harmonic, arithmetic, and geometric means of p and y as defined above.

Lemma 3.1: Let p and y be positive integers. If any two of the means t, h, and a of p and y are integers, then the third of these is also an integer. Moreover, h and t are of the same parity, as are p and y.

Proof: $h + t = \dfrac{2py}{p+y} + \dfrac{p^2 + y^2}{p+y} = \dfrac{(p+y)^2}{p+y} = p + y = 2a$. □

With the assistance of this lemma, it takes only a little experimentation to discover the mean crowds $(5, 45, 41, 9, 25, 15)$ and $(10, 40, 34, 16, 25, 20)$. The interested reader might wish to search for other examples before reading on.

Since each of the means A, G, H, and T is homogeneous [i.e., $A(kp, ky) = kA(p, y)$, etc.], it is immediate that any positive integral multiple of a mean crowd is also a mean crowd. Let us borrow a term associated with Pythagorean triples and say that a mean crowd is *primitive* if the greatest common divisor of its members is unity.

Conversely, it is clear (by the well-ordering principle) that any mean crowd must be some integral multiple of a primitive mean crowd; hence, the problem of characterizing mean crowds reduces to that of characterizing primitive mean crowds.

Lemma 3.2: The mean crowd (p, y, t, h, a, g) is primitive if and only if a and h are relatively prime.

Proof: The "if" part is obvious. For the "only if" part, let (p, y, t, h, a, g) be a mean crowd and suppose that q is a prime common divisor of a and h. By Lemma 3.1, $t = 2a - h$; therefore q divides t. By the Golden Proportionality, $ah = g^2$ so that q^2 divides g^2, and (since q is prime), q divides g.

Now $p + y = 2a$ and q divides the right side, so either q divides both p and y, or else q divides neither p nor y. But, by the Musical Proportionality, $ah = py$, so q^2 divides py, implying that q divides p, or q divides y. Hence, q divides both p and y.

Therefore, any prime common divisor of a and h also divides the other four members of the mean crowd (p, y, t, h, a, g), and the desired result follows. □

4. COMPLEMENTS OF MEAN CROWDS

Let us define the *complement* of the mean crowd (p, y, t, h, a, g) by

$$\text{comp}(p, y, t, h, a, g) = (a-g, a+g, a+h, a-h, a, a-p).$$

Lemma 4.1: The "complement" operator is involutory and preserves mean crowds and primitive mean crowds. That is, the complement of the complement of any mean crowd is the original mean crowd, and the complement of a mean crowd is a mean crowd, which is primitive if and only if the original is primitive.

Proof: Let us denote $\text{comp}(p, y, t, h, a, g)$ by (p', y', t', h', a', g'). Then we have

$$\begin{aligned}\text{comp}[\text{comp}(p, y, t, h, a, g)] &= \text{comp}(p', y', t', h', a', g')\\ &= (a'-g', a'+g', a'+h', a'-h', a', a'-p')\\ &= (a-(a-p), a+(a-p), a+(a-h), a-(a-h), a, a-(a-g))\\ &= (p, 2a-p, 2a-h, h, a, g) = (p, y, t, h, a, g).\end{aligned}$$

It is obvious that the members of the complement of a mean crowd are each positive integers. Moreover,

$$\begin{aligned}&A(a-g, a+g) = a = a',\\ &G(a-g, a+g) = \sqrt{(a-g)(a+g)} = \sqrt{a^2-g^2} = a-p = g',\\ &H(a-g, a+g) = \frac{(a-p)^2}{a} = \frac{(g')^2}{a'} = h', \quad \text{and}\\ &T(a-g, a+g) = 2a'-h' = 2a-(a-h) = a+h = t'.\end{aligned}$$

Therefore, the complement of a mean crowd is a mean crowd.

Now (p, y, t, h, a, g) is primitive if and only if $\text{GCD}(a, h) = 1$ (by Lemma 3.2), if and only if $\text{GCD}(a, a-h) = \text{GCD}(a', h') = 1$, if and only if (p', y', t', h', a', g') is primitive (again by Lemma 3.2). □

5. CHARACTERIZATIONS OF MEAN CROWDS

Characterization Theorem (Pythagorean Triple Form): Let p and y be positive numbers with $p < y$ and let a, g, h, and t be the arithmetic, geometric, harmonic, and tetra means, respectively, of p and y. Then the ordered sextuple (p, y, t, h, a, g) is a primitive mean crowd if and only if there exists a primitive Pythagorean triple $(u, v; w)$ such that $p = w^2 - wv$ and $y = w^2 + wv$.

Proof: (If) Let $(u, v; w)$ be a primitive Pythagorean triple with $p = w^2 - wv$ and $y = w^2 + wv$. Then

$$\begin{aligned}&a = w^2, \quad g = \sqrt{(w^2-wv)(w^2+wv)} = \sqrt{w^4 - w^2v^2} = \sqrt{w^2(w^2-v^2)} = wu,\\ &h = \frac{g^2}{a} = \frac{(wu)^2}{w^2} = u^2, \quad \text{and} \quad t = 2a - h = 2w^2 - u^2;\end{aligned}$$

each a positive integer, and therefore, (p, y, t, h, a, g) is a mean crowd. Moreover, $\text{GCD}(a, h) = \text{GCD}(w^2, u^2) = 1$ since $(u, v; w)$ is primitive; hence, (p, y, t, h, a, g) is primitive by Lemma 3.2.

(Only if) Let (p, y, t, h, a, g) be a primitive mean crowd. Then $\text{GCD}(a, h) = 1$ by Lemma 3.2. But $ah = g^2$ by the Golden Proportionality, so a and h are both squares. Also, by Lemma 4.1, $\text{comp}(p, y, t, h, a, g)$ is also a primitive mean crowd, and so, by the same reasoning, h' is also a square. Therefore, $(\sqrt{h}, \sqrt{a-h}; \sqrt{a})$ is a Pythagorean triple. Clearly, any common divisor of $\sqrt{a}$ and $\sqrt{h}$ is also a common divisor of a and h; hence, $(\sqrt{h}, \sqrt{a-h}; \sqrt{a})$ is primitive. □

This form of the Characterization Theorem allows for a striking illustration of the *complement* operator, as follows: The primitive mean crowd $(w^2 - wv, w^2 + wv, 2w^2 - u^2, u^2, w^2, wu)$ is generated by the primitive Pythagorean triple $(u, v; w)$; its complement is $(w^2 - wu, w^2 + wu, 2w^2 - v^2, v^2, w^2, wv)$, which is, of course, generated by the permuted triple $(v, u; w)$—quite literally a "complement."

Characterization Theorem (Sum of Squares Form): The ordered sextuple (p, y, t, h, a, g) is a primitive mean crowd if and only if there exist relatively prime integers α and β of opposite parity and $\beta < \alpha$ such that either $p = (\alpha - \beta)^2(\alpha^2 + \beta^2)$ and $y = (\alpha + \beta)^2(\alpha^2 + \beta^2)$ or $p = 2\beta^2(a^2 + \beta^2)$ and $y = 2\alpha^2(\alpha^2 + \beta^2)$.

Proof: The characterization theorem for primitive Pythagorean triples [7, p. 190] states that $(u, v; w)$ is a primitive Pythagorean triple with v even exactly when there exist relatively prime integers α and β of opposite parity with $\alpha > \beta > 0$ and $u = \alpha^2 - \beta^2$, $v = 2\alpha\beta$, and $w = \alpha^2 + \beta^2$. The desired result now follows immediately from the preceding theorem. □

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