REPRESENTING GENERALIZED LUCAS NUMBERS IN TERMS OF THEIR α-VALUES

Piero Filipponi

Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142 Rome, Italy

e-mail: filippo@fub.it

(*Submitted April 1997*)

1. INTRODUCTION

The elements of the sequences $\{A_k\}$ obeying the second-order recurrence relation

$$A_0,\ A_1 \text{ arbitrary real numbers, and } A_k = PA_{k-1} - QA_{k-2} \text{ for } k \geq 2 \tag{1.1}$$

are commonly referred to as *generalized Lucas numbers* with generating parameters P and Q (e.g., see [7], p. 41 ff.), and the equation

$$x^2 - Px + Q = 0 \tag{1.2}$$

is usually called the characteristic equation of $\{A_k\}$. In what follows, the root

$$\alpha_{P,Q} = (P + \sqrt{P^2 - 4Q})/2 \tag{1.3}$$

of (1.2) will be referred to as the *α-value* of $\{A_k\}$. Here, we shall confine ourselves to considering the well-known sequences $\{A_k\}$ that have $(A_0, A_1) = (2, P)$ or $(0, 1)$ as initial conditions, and

$$(P, Q) = (m, -1) \text{ or } (1, -m) \quad (m \text{ a natural number}) \tag{1.4}$$

as generating parameters (e.g., see [2] and [3]).

The aim of this note is to find the representation of the elements of these sequences in terms of their α-values. More precisely, we shall express A_k as

$$A_k = \sum_{r=-\infty}^{\infty} c_r \alpha_{P,Q}^r \quad [c_r = 0 \text{ or } m^{s(r)},\ s(r) \text{ nonnegative integers}] \tag{1.5}$$

with $c_r c_{r+1} = 0$, and with at most finitely many nonzero c_r.

The special case $m = 1$, for which, depending on the initial conditions, A_k equals the k^{th} Lucas number L_k or the k^{th} Fibonacci number F_k, is perhaps the most interesting (see Section 4).

2. REPRESENTING $V_k(m)$ AND $U_k(m)$

If we let $(P, Q) = (m, -1)$ in (1.1)-(1.3), then we get the numbers $V_k(m)$ and $U_k(m)$ (e.g., see [2]). They are defined by the second-order recurrence relation

$$A_k(m) = mA_{k-1}(m) + A_{k-2}(m) \tag{2.1}$$

(here A stands for either V or U) with initial conditions $V_0(m) = 2$, $V_1(m) = m$, $U_0(m) = 0$, and $U_1(m) = 1$. Their Binet forms are

$$V_k(m) = \alpha_m^k + \beta_m^k \quad \text{and} \quad U_k(m) = (\alpha_m^k - \beta_m^k)/\sqrt{m^2+4}, \tag{2.2}$$

where

$$\alpha_m := \alpha_{m,-1} = (m+\sqrt{m^2+4})/2 \tag{2.3}$$

is the α-value of (2.1) and

$$\beta_m = -1/\alpha_m = m-\alpha_m. \tag{2.3'}$$

Observe that $V_k(1) = L_k$ and $U_k(1) = F_k$, whereas $V_k(2) = Q_k$ (the k^{th} Pell-Lucas number) and $U_k(2) = P_k$ (the k^{th} Pell number) (e.g., see [5]).

The α-representations of $V_k(m)$ and $U_k(m)$ are presented in Subsection 3.1 below and then proved in detail in Subsection 3.2.

2.1 Results

$$V_{2k}(m) = \alpha_m^{-2k} + \alpha_m^{2k} \quad (k = 0, 1, 2, \ldots). \tag{2.4}$$

Remark 1: For $k = 0$ the r.h.s. of (2.4) is correct but it is not the α-representation of $V_0(m)$.

$$V_{2k+1}(m) = \sum_{r=1}^{2k+1} m\alpha_m^{2r-2(k+1)} \quad (k = 0, 1, 2, \ldots), \tag{2.5}$$

$$U_{2k}(m) = \sum_{r=1}^{k} m\alpha_m^{4r-2(k+1)} \quad (k = 1, 2, 3, \ldots), \tag{2.6}$$

$$U_{2k+1}(m) = \alpha_m^{-2k} + \sum_{r=1}^{k} m\alpha_m^{4r-2k-1} \quad (k = 0, 1, 2, \ldots). \tag{2.7}$$

Remark 2: Under the usual assumption that a sum vanishes whenever the upper range indicator is less than the lower one, (2.7) applies also for $k = 0$.

2.2 Proofs

The proof of (2.4) can be obtained trivially by using (2.2) and (2.3').

Proof of (2.5): Use the geometric series formula (g.s.f.) and (2.2)-(2.3') to rewrite the r.h.s. of (2.5) as

$$\frac{m\alpha_m^{-2k}(\alpha_m^{4k+2}-1)}{\alpha_m^2-1} = \frac{m\alpha_m}{\alpha_m^2-1}[\alpha_m^{2k+1} - \alpha_m^{-(2k+1)}] = \alpha_m^{2k+1} - \alpha_m^{-(2k+1)} = V_{2k+1}(m). \quad \square$$

Proof of (2.6): Use the g.s.f. and (2.2)-(2.3') to rewrite the r.h.s. of (2.6) as

$$\frac{m\alpha_m^{2-2k}(\alpha_m^{4k}-1)}{\alpha_m^4-1} = \frac{m\alpha_m(\alpha_m^{2k+1}-\alpha_m^{1-2k})}{(\alpha_m^2+1)(\alpha_m^2-1)} = \frac{\alpha_m^{2k+1}-\alpha_m^{1-2k}}{\alpha_m^2+1}$$

$$= \frac{\alpha_m(\alpha_m^{2k}-\alpha_m^{-2k})}{\alpha_m\sqrt{m^2+4}} = U_{2k}(m). \quad \square$$

To prove (2.7), we need the identity

$$\alpha_m U_n(m) + (-1)^n \alpha_m^{-n} = U_{n+1}(m). \tag{2.8}$$

Proof of (2.8): Use (2.2), (2.3'), and the relation $\sqrt{m^2+4} = \alpha_m - \beta_m$ to rewrite the l.h.s. of (2.8) as

$$\frac{\alpha_m(\alpha_m^n-\beta_m^n)}{\sqrt{m^2+4}}+\beta_m^n=\frac{\alpha_m^{n+1}+\beta_m^{n-1}}{\sqrt{m^2+4}}+\beta_m^n=\frac{\alpha_m^{n+1}+\beta_m^{n-1}(\beta_m\sqrt{m^2+4}+1)}{\sqrt{m^2+4}}$$

$$=\frac{\alpha_m^{n+1}+\beta_m^{n-1}(-\beta_m^2)}{\sqrt{m^2+4}}=U_{n+1}(m). \quad \square$$

Proof of (2.7): Use the g.s.f., (2.2)-(2.3′), and (2.8) to rewrite the r.h.s. of (2.7) as

$$\alpha_m^{-2k}+\frac{m\alpha_m^{3-2k}(\alpha_m^{4k}-1)}{\alpha_m^4-1}=\alpha_m^{-2k}+\frac{m\alpha_m(\alpha_m^{2k+2}-\alpha_m^{2-2k})}{(\alpha_m^2+1)(\alpha_m^2-1)}=\alpha_m^{-2k}+\frac{\alpha_m^{2k+2}-\alpha_m^{2-2k}}{\alpha_m^2+1}$$

$$=\alpha_m^{-2k}+\frac{\alpha_m^2(\alpha_m^{2k}-\alpha_m^{-2k})}{\alpha_m\sqrt{m^2+4}}=\alpha_m^{-2k}+\alpha_m U_{2k}(m)=U_{2k+1}(m). \quad \square$$

3. REPRESENTING $H_k(m)$ AND $G_k(m)$

If we let $(P,Q)=(1,-m)$ in (1.1)-(1.3), then we get the numbers $H_k(m)$ and $G_k(m)$ (e.g., see [3]). They are defined by the second-order recurrence relation

$$A_k(m)=A_{k-1}(m)+mA_{k-2}(m) \tag{3.1}$$

(here A stands for H or G) with initial conditions $H_0(m)=2$, $H_1(m)=G_1(m)=1$, and $G_0(m)=0$. Their Binet forms are

$$H_k(m)=\gamma_m^k+\delta_m^k \quad\text{and}\quad G_k(m)=(\gamma_m^k-\delta_m^k)/\sqrt{4m+1}, \tag{3.2}$$

where

$$\gamma_m:=\alpha_{1,-m}=(1+\sqrt{4m+1})/2 \tag{3.3}$$

is the α-value of (3.1), and

$$\delta_m=-m/\gamma_m=1-\gamma_m. \tag{3.3′}$$

Observe that $H_k(1)=V_k(1)=L_k$ and $G_k(1)=U_k(1)=F_k$, whereas $H_k(2)=j_k$ (the k^{th} Jacobsthal-Lucas number) and $G_k(2)=J_k$ (the k^{th} Jacobsthal number) (see [6]).

The α-representations of $H_k(m)$ and $G_k(m)$ are shown in Subsection 3.1 below. To save space, only identity (3.7) will be proved in detail in Subsection 3.2.

3.1 Results

$$H_{2k}(m)=m^{2k}\gamma_m^{-2k}+\gamma_m^{2k}\quad (k=0,1,2,\ldots;\text{ see Remark 1}), \tag{3.4}$$

$$H_{2k+1}(m)=\sum_{r=1}^{2k+1}m^{2k+1-r}\gamma_m^{2r-2(k+1)}\quad (k=0,1,2,\ldots), \tag{3.5}$$

$$G_{2k}(m)=\sum_{r=1}^{k}m^{2(k-r)}\gamma_m^{4r-2(k+1)}\quad (k=1,2,3,\ldots), \tag{3.6}$$

$$G_{2k+1}(m)=m^{2k}\gamma_m^{-2k}+\sum_{r=1}^{k}m^{2(k-r)}\gamma_m^{4r-2k-1}\quad (k=0,1,2,\ldots;\text{ see Remark 2}). \tag{3.7}$$

A Special Case (cf. (2.3) and (2.4) of [6]): For $m=2$ $(=\gamma_2)$, identities (3.4)-(3.7) reduce to

$$H_{2k}(2)=4^k+1=j_{2k}, \tag{3.4$'$}$$

$$H_{2k+1}(2)=\sum_{r=1}^{2k+1}2^{r-1}=2^{2k+1}-1=j_{2k+1} \tag{3.5$'$}$$

$$G_{2k}(2)=\sum_{r=1}^{k}2^{2(r-1)}=(4^k-1)/3=J_{2k}, \tag{3.6$'$}$$

$$G_{2k+1}(2)=1+\sum_{r=1}^{k}2^{2r-1}=(2^{2k+1}+1)/3=J_{2k+1}. \tag{3.7$'$}$$

3.2 A Proof

To prove (3.7), we need the identity

$$\gamma_m G_n(m)+\delta_m^n=G_{n+1}(m). \tag{3.8}$$

Proof of (3.8): Use (3.2) and the relation $\sqrt{4m+1}=\gamma_m-\delta_m$ to rewrite the l.h.s. of (3.8) as

$$\frac{\gamma_m(\gamma_m^n-\delta_m^n)}{\sqrt{4m+1}}+\delta_m^n=\frac{\gamma_m^{n+1}-\delta_m^n(\gamma_m-\sqrt{4m+1})}{\sqrt{4m+1}}=\frac{\gamma_m^{n+1}-\delta_m^n\delta_m}{\sqrt{4m+1}}=G_{n+1}(m).\ \square$$

Proof of (3.7): Use the g.s.f., (3.2)-(3.3$'$), and (3.8) to rewrite the r.h.s. of (3.7) as

$$\begin{aligned}
m^{2k}\gamma_m^{-2k}+m^{2(k-1)}\gamma_m^{3-2k}\sum_{s=0}^{k-1}(\gamma_m^4/m^2)^s &= m^{2k}\gamma_m^{-2k}+m^{2(k-1)}\gamma_m^{3-2k}\frac{(\gamma_m^4/m^2)^k-1}{\gamma_m^4/m^2-1}\\
&= m^{2k}\gamma_m^{-2k}+\frac{\gamma_m^{2k+3}-m^{2k}\gamma_m^{3-2k}}{\gamma_m^4-m^2}=m^{2k}\gamma_m^{-2k}+\frac{\gamma_m^{2k+3}-m^{2k}\gamma_m^{3-2k}}{\gamma_m^2\sqrt{4m+1}}\\
&= m^{2k}\gamma_m^{-2k}+\frac{\gamma_m^{2k+1}-m^{2k}\gamma_m^{1-2k}}{\sqrt{4m+1}}=\delta_m^{2k}+\frac{\gamma_m^{2k+1}-\delta_m^{2k}\gamma_m}{\sqrt{4m+1}}\\
&= \delta_m^{2k}+\gamma_m G_{2k}(m)=G_{2k+1}(m).\ \square
\end{aligned}$$

4. A REMARKABLE CASE ($m=1$)

The *β-expansion* of any natural number N is the *unique* finite sum of distinct integral powers of the golden section α that equals N and contains no consecutive powers of α. This expansion was first studied by Bergman in [1] where the author used the symbol β instead of α to denote the golden section.

As already mentioned in the previous sections, if we let $m=1$ in (2.1)-(2.3) [or in (3.1)-(3.3)], then we get either the Lucas numbers L_k or the Fibonacci numbers F_k, depending on the initial conditions of the recurrence relation (2.1) [or (3.1)] whose α-value

$$\alpha:=\alpha_1=\gamma_1=(1+\sqrt{5})/2 \tag{4.1}$$

is the golden section. Consequently, if we let $m=1$ in (1.5), then it is evident that $c_r\in\{0,1\}$ so that letting $m=1$ in (2.4)-(2.7) [or in (3.4)-(3.7)] yields the β-expansions of L_k and F_k. As an illustration, from (2.5) [or (3.5)], we see that the β-expansion of L_{2k+1} is

$$L_{2k+1}=\alpha^{-2k}+\alpha^{-2k+2}+\cdots+\alpha^0+\cdots+\alpha^{2k-2}+\alpha^{2k}. \tag{4.2}$$

Remark 3: By letting $m=1$ in (2.4), one gets $L_{2k}=\alpha^{-2k}+\alpha^{2k}$. This expression works correctly also for $k=0$ but, in this case, it is not the β-expansion of L_0, as stated in Remark 1. In fact, this expansion is $L_0=2=2\alpha^{-1}\alpha=\alpha^{-1}(1+\sqrt{5})=\alpha^{-1}(\alpha+\alpha^{-1}+1)=\alpha^{-1}(\alpha^2+\alpha^{-1})=\alpha^{-2}+\alpha$.

5. CONCLUDING COMMENTS

The representations established in this note for certain generalized Lucas numbers, besides being of some interest *per se*, allow us to derive some cute identities involving them. For example, by using (2.5), (2.6), and (2.4), we get

$$\frac{V_{2k+1}(m)}{m}=1+\sum_{r=1}^{k}V_{2r}(m), \tag{5.1}$$

$$\frac{U_{2k}(m)}{m}=\begin{cases}\sum_{r=1}^{k/2}V_{4r-2}(m) & (k \text{ even}),\\ 1+\sum_{r=1}^{(k-1)/2}V_{4r}(m) & (k \text{ odd}),\end{cases} \tag{5.2}$$

$$\frac{V_{2k+1}^2(m)}{m^2}=2k+1+\sum_{r=1}^{2k}rV_{4k+2-2r}(m), \tag{5.3}$$

whereas, from (3.5) and (3.4), we obtain

$$H_{2k+1}(m)=m^k+\sum_{r=1}^{k}m^{k-r}H_{2r}. \tag{5.4}$$

The most interesting among the above identities is, perhaps, the identity (5.3) which, for $m=1$, gives a rather unusual expression for the squares of odd-subscripted Lucas numbers. Let us give a sketch of its proof.

Proof of (5.3) (a sketch): Use (2.5) and (2.4) to write

$$\begin{aligned}V_{2k+1}^2(m)&=m^2[\alpha_m^{-4k}+2\alpha_m^{-4k+2}+3\alpha_m^{-4k+4}+\cdots+(2k+1)\alpha_m^0+\cdots+3\alpha_m^{4k-4}+2\alpha_m^{4k-2}+\alpha_m^{4k}]\\&=m^2[V_{4k}(m)+2V_{4k-2}(m)+\cdots+2kV_2(m)+2k+1].\ \square\end{aligned}$$

Using the same technique led us to discover a quite amazing expression for the cubes of odd-subscripted Lucas numbers. Namely, we get

$$L_{2k+1}^3=\sum_{r=1}^{2k+1}T_rL_{6k+2-2r}+\sum_{r=1}^{k-1}(S_k-r^2)L_{2r}+S_k \quad (k\geq 1), \tag{5.5}$$

where T_k denotes the k^{th} triangular number and $S_k=3k(k+1)+1$. A direct proof of (5.5) can be carried out by using expressions for $\sum_r L_{a+hr}$, $\sum_r rL_{a+hr}$, and $\sum_r r^2L_{a+hr}$, the last two of which can be obtained from the Binet form for Lucas numbers and (3.1)-(3.2) of [4].

ACKNOWLEDGMENTS

This work has been carried out within the framework of an agreement between the Italian PT Administration (Istituto Superiore PT) and the Fondazione Ugo Bordoni.

REFERENCES

1. G. Bergman. "A Number System with an Irrational Base." *Math. Magazine* **31** (1957):98-110.
2. L. A. G. Dresel. "On Pseudoprimes Related to Generalized Lucas Sequences." *The Fibonacci Quarterly* **35.1** (1997):35-42.
3. P. Filipponi. "A Note on a Class of Lucas Sequences." *The Fibonacci Quarterly* **29.3** (1991):256-63.
4. P. Filipponi & A. F. Horadam. "Derivative Sequences of Fibonacci and Lucas Polynomials." In *Applications of Fibonacci Numbers* **4**:99-108. Dordrecht: Kluwer, 1991.
5. A. F. Horadam & Bro. J. M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985):7-20.
6. A. F. Horadam. "Jacobsthal Representation Numbers." *The Fibonacci Quarterly* **34.1** (1996):40-54.
7. P. Ribenboim. *The Book of Prime Number Records.* New York: Springer-Verlag, 1988.

AMS Classification Numbers: 11D85, 11B39, 11A63

❖❖❖