MORGAN-VOYCE TYPE GENERALIZED POLYNOMIALS WITH NEGATIVE SUBSCRIPTS

A. F. Horadam

The University of New England, Armidale, Australia 2351

(*Submitted January 1997*)

1. PURPOSE OF THE PAPER

Previous papers ([1], [2], [3], and [4]) have investigated aspects of the Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$ and certain polynomials $C_n(x)$, $c_n(x)$ associated with them, together with generalizations of them, $P_n^{(r)}(x)$, $Q_n^{(r)}(x)$, namely,

$$P_{n-1}^{(0)}(x) = b_n(x), \tag{1.1}$$

$$P_{n-1}^{(1)}(x) = B_n(x), \tag{1.2}$$

$$P_{n-1}^{(2)}(x) = c_n(x), \tag{1.3}$$

$$Q_n^{(0)}(x) = C_n(x). \tag{1.4}$$

Both generalizations are absorbed into a *composite* polynomial $R_n^{(r,u)}(x)$ such that [4]

$$R_n^{(r,1)}(x) = P_n^{(r)}(x), \tag{1.5}$$

$$R_n^{(r,2)}(x) = Q_n^{(r)}(x). \tag{1.6}$$

Here we consider the implications for the theory in the case $R_{-n}^{(r,u)}(x)$, where $n > 0$.

Because of the detailed information in the previous papers, only the algebraic skeletal structure of the new system of polynomials will be outlined.

For the record, we list the following equalities involving negative subscripts which are readily obtainable from the Binet forms in [2]:

$$B_{-n}(x) = -B_n(x), \tag{1.7}$$

$$b_{-n}(x) = b_{n+1}(x), \tag{1.8}$$

$$C_{-n}(x) = C_n(x), \tag{1.9}$$

$$c_{-n}(x) = -c_{n+1}(x). \tag{1.10}$$

Additionally, we require

$$P_n^{(r)}(1) = F_{2n+1} + rF_{2n} \qquad [1], \tag{1.11}$$

$$Q_n^{(r)}(1) = L_{2n} + rF_{2n} \qquad [3], \tag{1.12}$$

$$Q_n^{(2r+1)}(1) = 2P_n^{(r)}(1) \qquad [3], \tag{1.13}$$

$$Q_n^{(r)}(x) = P_n^{(r)}(x) + P_{n-1}^{(0)}(x) \quad (n \geq 1) \quad [3], \tag{1.14}$$

whence

$$Q_n^{(r)}(1) - P_n^{(r)}(1) = F_{2n-1}. \tag{1.15}$$

Worth recording finally is ([2], (1.7), (1.9)) the differential equation

$$\frac{dC_{-n}(x)}{dx} = -nB_{-n}(x). \tag{1.16}$$

2. THE POLYNOMIALS $R_{-n}^{(r,u)}(x)$

Define the polynomials $\{R_{-n}^{(r,u)}(x)\}$ by means of a Morgan-Voyce type recurrence

$$R_{-n}^{(r,u)}(x) = (x+2)R_{-n-1}^{(r,u)}(x) - R_{-n-2}^{(r,u)}(x) \quad (n>0) \tag{2.1}$$

with

$$R_0^{(r,u)}(x) = u, \quad R_{-1}^{(r,u)}(x) = (u-1)x + u - r. \tag{2.2}$$

Paralleling the data in [4], we postulate the existence of a sequence of integers $\{c_{-n,k}^{(r,u)}\}$, $n \geq 0$, for which

$$R_{-n}^{(r,u)}(x) = \sum_{k=0}^{n} c_{-n,k}^{(r,u)} x^k, \tag{2.3}$$

in which

$$c_{-n,n}^{(r,u)} = \begin{cases} u, & n=0, \\ u-1, & n>0, \end{cases} \tag{2.4}$$

and

$$c_{-n,0}^{(r,u)} = u - nr. \tag{2.5}$$

Moreover, for $x=0$ in (2.1) and (2.3),

$$c_{-n,0}^{(r,u)} = 2c_{-n-1,0}^{(r,u)} - c_{-n-2,0}^{(r,u)}. \tag{2.6}$$

Furthermore, (2.1) leads to ($k \geq 1$)

$$c_{-n,k}^{(r,u)} = 2c_{-n-1,k}^{(r,u)} - c_{-n-2,k}^{(r,u)} + c_{-n-1,k-1}^{(r,u)}. \tag{2.7}$$

The Coefficients $c_{-n,k}^{(r,u)}$

Repeated use of (2.1) and (2.2) allows us to construct a table of the coefficients $c_{-n,k}^{(r,u)}$ as follows.

TABLE 1. The Coefficients $c_{-n,k}^{(r,u)}$ $(n \geq 0)$

$n \backslash k$	0	1	2	3	4	5	6
0	u						
-1	$u-r$	$-1+u$					
-2	$u-2r$	$-2+3u-r$	$-1+u$				
-3	$u-3r$	$-3+6u-4r$	$-4+5u-r$	$-1+u$			
-4	$u-4r$	$-4+10u-10r$	$-10+15u-6r$	$-6+7u-r$	$-1+u$		
-5	$u-5r$	$-5+15u-20r$	$-20+35u-21r$	$-21+28u-8r$	$-8+9u-r$	$-1+u$	
-6	$u-6r$	$-6+21u-35r$	$-35+70u-56r$	$-56+84u-36r$	$-36+45u-10r$	$-10+11u-r$	$-1+u$

Comparison of this table with the corresponding table for $c_{n,k}^{(r,u)}$ in [4] reveals that the sign of the constants and the sign of the coefficients for r have both changed from + to −. On the other hand, the sign of the coefficients of u remains unchanged (+), but n has been replaced by $-n+1$. That is, from [4], we have the key formula

$$c_{-n,k}^{(r,u)} = -\binom{n+k-1}{2k-1} - r\binom{n+k}{2k+1} + u\binom{n+k}{2k} \tag{2.8}$$

$$= \binom{n+k-1}{2k} - r\binom{n+k}{2k+1} + (u-1)\binom{n+k}{2k}, \tag{2.9}$$

by Pascal's Theorem.

Suitable specializations $u = 1, 2$ in (1.5) and (1.6) reduce this to

$$a_{-n,k}^{(r)} = \binom{n+k-1}{2k} - r\binom{n+k}{2k+1} \tag{2.10}$$

and

$$b_{-n,k}^{(r)} = \binom{n+k}{2k} + \binom{n+k-1}{2k} - r\binom{n+k}{2k+1} \tag{2.11}$$

for $P_{-n}^{(r)}(x)$ and $Q_{-n}^{(r)}(x)$, respectively, tables for which the reader may care to construct.

Further specializations are obvious, e.g., $a_{-n,k}^{(0)} = \binom{n+k-1}{2k}$.

Next, multiply (2.9) throughout by x^k and sum. Then, by (1.5) $(n \to -n)$, (1.8), and (2.3), we deduce that

Theorem 1: $R_{-n}^{(r,u)}(x) = P_{-n}^{(r)}(x) + (u-1)b_{n+1}(x)$.

Numerical Specializations

Using (1.1)-(1.14) variously, we deduce that

$$R_{-n}^{(0,1)}(1) = P_{-n}^{(0)}(1) = b_n(1) = F_{2n-1}, \tag{2.12}$$

$$R_{-n}^{(1,1)}(1) = P_{-n}^{(1)}(1) = -B_{n-1}(1) = -F_{2n-2}, \tag{2.13}$$

$$R_{-n}^{(2,1)}(1) = P_{-n}^{(2)}(1) = -c_n(1) = -L_{2n-1}, \tag{2.14}$$

$$R_{-n}^{(0,2)}(1) = Q_{-n}^{(0)}(1) = C_n(1) = L_{2n}, \tag{2.15}$$

$$R_{-n}^{(1,2)}(1) = Q_{-n}^{(1)}(1) = 2P_{-n}^{(0)}(1) = 2b_n(1) = 2F_{2n-1}, \tag{2.16}$$

$$R_{-n}^{(2,2)}(1) = Q_{-n}^{(2)}(1) = F_{2n+3}. \tag{2.17}$$

Also [cf. (2.13)],

$$R_{-n}^{(0,0)}(1) = B_{-n}(1) = F_{2n}. \tag{2.18}$$

Moreover, we have from (2.1) that

$$R_{-n-1}^{(r,u)}(-1) = R_{-n}^{(r,u)}(-1) + R_{-n-2}^{(r,u)}(-1), \tag{2.19}$$

$$R_{-n-1}^{(r,u)}(-3) = -(R_{-n}^{(r,u)}(-3) + R_{-n-2}^{(r,u)}(-3)), \tag{2.20}$$

$$R_{-n}^{(r,u)}(-2) = -R_{-n}^{(r,u)}(-2) \tag{2.21}$$

$\left[\text{e.g., } R_{-1}^{(r,u)}(-2) = -u-r+2 = -R_{-3}^{(r,u)}(-2)\right]$.

3. MISCELLANEOUS RESULTS

Chebyshev Polynomials

Employing the notation in [4] for the Chebyshev polynomials $U_n(x)$ and $T_n(x)$, we discover that, with (1.7)-(1.10),

$$B_{-n}(x) = -U_n\left(\frac{x+2}{2}\right), \tag{3.1}$$

$$C_{-n}(x) = 2T_n\left(\frac{x+2}{2}\right), \tag{3.2}$$

$$b_{-n}(x) = U_{n+1}\left(\frac{x+2}{2}\right) - U_n\left(\frac{x+2}{2}\right), \tag{3.3}$$

$$c_{-n}(x) = -U_{n+1}\left(\frac{x+2}{2}\right) + U_n\left(\frac{x+2}{2}\right). \tag{3.4}$$

[Ordinarily, $U_{-n}(x) = -U_{n-2}(x)$, but this is not true when x is replaced by $\frac{x+2}{2}$.]

As in [4], we have

Theorem 2: $R_{-n}^{(r,u)}(x) = -B_{-n-1}(x) - (r+u-2)B_{-n}(x) + (u-1)B_{-n+1}(x)$.

Theorem 3: $R_{-n}^{(r,u)}(x) = ((u-1)x - r + u)B_{-n}(x) - uB_{-n-1}$.

Both these theorems can, by (3.1), be cast in terms of $U_n(\frac{x+2}{2})$. Theorem 3 is, in fact, an equivalent of the Binet form for $R_{-n}(x)$. A Simson formula analog for $R_{-n}(x)$ corresponding to that in [4] for $R_n(x)$ is left to the reader's interest, and likewise for a generating function analog.

Zeros and Orthogonality

These properties for $B_{-n}(x), \ldots, c_{-n}(x)$ may be approached as for those of $B_n(x), \ldots, c_n(x)$ in [2], by referring to (1.7)-(1.10).

Rising Diagonals

Rising diagonal polynomials (functions) are obtained from Table 1 by considering a set of upward-slanting parallel diagonal lines (cf. [2]). Designate these polynomials by $\mathfrak{R}_{-n}^{(r,u)}(x)$ or just $\mathfrak{R}_{-n}(x)$ for brevity. Then $\mathfrak{R}_0(x) = u - r$, $\mathfrak{R}_{-1}(x) = u - 2r + (u-1)x$.

A little tricky exploration enables us to affirm that [see (2.9)]

$$\mathfrak{R}_{-n}(x) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} c_{-n-1+k,\,k}x^k. \tag{3.5}$$

Comparison with [2, (7.1)] is worthwhile at this point. The contrast in the two forms demonstrates that, in passing from $\mathfrak{R}_n(x)$ in [2] to $\mathfrak{R}_{-n}(x)$ here, we cannot with impunity always merely replace n by its negative. Asymmetry in the two patterns of rising diagonals explains this dilemma. [Indeed, $\mathfrak{R}_0(x)$ is chosen to be different in [2] and here.]

Adopting [2] as our model, we are able to establish the following corresponding results (no proofs offered.)

Theorem 4 (Recurrence): $\mathfrak{R}_{-n}(x) = 2\mathfrak{R}_{-n+1}(x) + (x-1)\mathfrak{R}_{-n+2}(x)$.

Corollary 1: $\mathfrak{R}_{-n}(1) = 2^{n-1}\{2u - 2r - 1\}$.

Theorem 5 (Generating function):

$$\sum_{i=0}^{\infty} \mathfrak{R}_{-i}(x) y^i = \{u - r + [-u + x(u-1)]y\}\{1 - (2y + (x-1)y^2)\}^{-1}.$$

Analogously to the procedures in [2], we may derive a Binet form and a Simson formula for $\mathfrak{R}_{-n}(x)$.

4. CONCLUSION

The development outlined above complements that in [4] and thus rounds out the general theory for integer n (about which more could be written).

REFERENCES

1. R. André-Jeannin. "A Generalization of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **32.3** (1994):228-31.
2. A. F. Horadam. "New Aspects of Morgan-Voyce Polynomials." In *Applications of Fibonacci Numbers* **7:**161-76. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1998.
3. A. F. Horadam. "Polynomials Associated with Generalized Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **34.4** (1996):342-48.
4. A. F. Horadam. "A Composite of Morgan-Voyce Polynomials." *The Fibonacci Quarterly* **35.3** (1997):233-39.

AMS Classification Number 11B39

❖❖❖

ACKNOWLEDGMENT OF PRIORITY

It has been brought to my attention by Dr. John Holte by way of Dr. Bergum that there was a failure to give a "complete list of references" in my article "The Fibonacci Triangle Modulo p" (June-July 1998 issue of *The Fibonacci Quarterly*). My research was performed in spring and summer of 1995. The paper did not appear until 1998 because it references an unpublished paper of Dr. William Webb and Dr. Diana Wells that Dr. Bergum asked me to get permission to cite. My research therefore post-dates Dr. Holte's article "A Lucas-Type Theorem for Fibonomial Coefficient Residues" (February 1994 issue of *The Fibonacci Quarterly*) of which I was unaware until after the publication of my article. While the results were obtained independently and without knowledge of Dr. Holte's work, Dr. Holte has asked that I give an acknowledgment of priority. I acknowledge that Dr. Holte has priority for any results common to the two papers. As a final note, I would like to add that the starting point for my research was a paper by Dr. Diana Wells, "The Fibonacci and Lucas Triangle Modulo 2" (April 1994 issue of *The Fibonacci Quarterly*) which also failed to reference Dr. Holte's paper and contains results that Holte claims priority for in his letter to Dr. Bergum.

Brad Wilson