RECURRENCE RELATIONS FOR POWERS OF RECURSION SEQUENCES

Adam Michael Stinchcombe
Adirondack Community College, 640 Bay Road, Queensbury, NY 12804
(*Submitted February 1997*)

1. INTRODUCTION

In this article I present some partial answers to the open questions raised by Cooper and Kennedy in [1]. In that article the authors asked whether there exists a recurrence relation among powers x_n^l, where x_n represents the solution to a given recurrence relation. I answer this in the affirmative below, include a few details about the corresponding order, then indicate a way to calculate any such relation the reader might seek, and, finally, state a few results from such calculations.

An informal sketch of the proof and procedure runs as follows. Every solution to a recurrence relation can be expressed as a linear combination of powers of roots to the characteristic polynomial. The coefficients of the original recurrence relation are the elementary symmetric polynomials in these roots. Every power of a solution can be expressed as a linear combination of products of powers of these roots by using the general multinomial theorem. These products can be used as roots to form a new characteristic polynomial. On inspection, the coefficients of this new characteristic polynomial are symmetric in the roots of the old characteristic polynomial, and, therefore, can be expressed as polynomials in the elementary symmetric polynomials of the roots; that is, the coefficients of the new recurrence relation can be expressed in terms of the coefficients of the original characteristic polynomial. There is a method for obtaining the expression, amounting to a multivariate version of the Euclidean algorithm.

2. EXISTENCE

Let $x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}$ be a linear homogeneous recurrence relation with constant coefficients $\{a_i \mid i = 1, \ldots, k\}$ and of order k. Let $p(x) = x^k - a_1x^{k-1} - \cdots - a_k$ be the characteristic polynomial for this relation. Let $p(x)$ factor as $p(x) = (x - r_1)(x - r_2)\cdots(x - r_k)$ over the field of complex numbers and suppose that the roots are distinct. We can write the Binet closed form for x_n as $x_n = A_1r_1^n + A_2r_2^n + \cdots + A_kr_k^n$. The constants $\{A_i \mid i = 1, \ldots, k\}$ are determined by the initial conditions specified in a particular solution to the recurrence relation.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ be two k-tuples. Define the symbol α^β to be the product of all terms α_i raised to the β_i power:

$$\alpha^\beta = \prod_{i=1}^{k} (\alpha_i^{\beta_i}).$$

Writing $X_n = (A_1r_1^n, A_2r_2^n, \ldots, A_kr_k^n)$, $A = (A_1, A_2, \ldots, A_k)$, and $R = (r_1, r_2, \ldots, r_k)$, we see that $X^\alpha = A^\alpha(R^\alpha)^n$ for each k-tuple α.

Recall the definition of the multinomial coefficient $c(\alpha) = (\alpha_1 + \cdots + \alpha_k)!/(\alpha_1! \ldots \alpha_k!)$.

Introduce the indexing set $B_l = \{(i_1, \ldots, i_k) \mid \text{each } i_j \text{ is a nonnegative integer and } i_1 + \cdots + i_k = l\}$.

Theorem 1: If $x_n = \sum_{i=1}^{k} A_i r_i^n$, then $x_n^l = \sum_{\alpha \in B_l} c(\alpha) A^\alpha (R^\alpha)^n$.

Proof:

$$x_n^l = \left(\sum_{i=1}^{k} A_i r_i^n \right)^l = \sum_{\alpha \in B_l} c(\alpha) X^\alpha = \sum_{\alpha \in B_l} c(\alpha) A^\alpha (R^\alpha)^n$$

by the general multinomial Theorem (see Hungerford [3, Th. 1.6, p. 118]). □

Therefore, $y_n = x_n^l$ is a linear combination of terms that are products of roots from the original characteristic polynomial of total degree l, raised to the n^{th} power. Thus, y_n is a solution to a recurrence relation. The next theorem tells us more.

Theorem 2: The characteristic polynomial for the sequence y_n can be written in terms of the coefficients of the characteristic polynomial $p(x)$ for x_n.

Proof: The characteristic polynomial for y_n is

$$q(x) = \prod_{\alpha \in B_l} (x - R^\alpha) \in \mathbb{C}[x] \text{ by Theorem 1.}$$

Consider some permutation $\sigma: \{r_1, \ldots, r_k\} \to \{r_1, \ldots, r_k\}$ of the roots of $p(x)$. σ can be decomposed into a product of transpositions [3, p. 48]. Each transposition interchanges two roots, say r_m and r_n. The effect of this transposition is to interchange the exponents from $\alpha = (i_1, \ldots, i_m, \ldots, i_n, \ldots, i_k)$ to $\alpha' = (i_1, \ldots, i_n, \ldots, i_m, \ldots, i_k)$ within the indexing set B_l. Thus, each transposition represents a transposition of the elements of B_l because the conditions defining k-tuples in B_l are unchanged by switching values positionally. The composition of transpositions that give σ also describe a composition of transpositions in B_l. Thus, σ gives rise to a permutation of B_l. Since the product for $q(x)$ is formed over the entire set B_l, this permutation leaves $q(x)$ fixed.

If we were to expand $q(x)$ into its standard form, the coefficients would be polynomial expressions in the roots $\{r_1, \ldots, r_k\}$. These coefficients are invariant under permutation of the roots and so are symmetric polynomials in the roots. Any such symmetric polynomial can be expressed as a polynomial in the elementary symmetric functions [2, p. 307].

Since these elementary symmetric polynomials are exactly the coefficients of the characteristic polynomial $p(x)$, the coefficients of $q(x)$ can be written as expressions in the coefficients of $p(x)$. □

3. ORDER

What is the order of the recurrence relation $y_n = x_n^l$?

It should be the degree of the characteristic polynomial $q(x)$. This degree is counted by the number of elements in B_l. Given a value of k, define $S(k, 1) = |B_l|$ where, recall, $B_l = \{(i, \ldots, i_k) |$ each i_j is a nonnegative integer and $i_1 + \cdots + i_k = l\}$.

Theorem 3: $S(k, l)$ obeys the relations: $S(k, 1) = k$ for all k, $S(1, l) = l$ for all l, and $S(k, l) = S(k-1, l) + S(k, l-1)$ for every k and l.

Proof: Proceed inductively.

$S(k, 1)$ represents the number of ways to define a k-tuple of nonnegative integers that add to 1. There are k ways to do this, corresponding to placing a 1 in any one of the k places in the k-tuple and 0 everywhere else.

$S(1, l)$ represents the number of ways to have a 1-tuple of nonnegative integers that add to l. There is only one such way.

Let $\alpha = (i_1, \dots, i_k)$ be a generic element of B_l. Either $i_1 = 0$ or $i_1 > 0$. If $i_1 = 0$, then $i_2 + \cdots + i_k = l$, and the number of possible ways to select such i_j's is $S(k-1, l)$; in other words, a sum of l obtained with $k-1$ variables. If $i_1 > 0$, then we can subtract one from i_1 to get $(i_1 - 1, \dots, i_k)$ as an element of B_{l-1} that has $S(k, l-1)$ elements. Therefore, the total number of possibilities is $S(k-1, l) + S(k, l-1)$. □

Theorem 4: $S(k, l) = {}_{l+k-1}C_{k-1}$, where ${}_iC_j$ stands for the binomial coefficient $i!/[j!(i-j)!]$.

Proof: Substituting $l = 1$ gives ${}_{l+k-1}C_{k-1} = {}_kC_{k-1} = k$ while substituting $k = 1$ gives ${}_{l+k-1}C_{k-1} = {}_lC_0 = 1$. To check that the given binomial coefficient satisfies the recurrence relation, simplify

$${}_{l+(k-1)-1}C_{(k-1)-1} + {}_{(l-1)+k-1}C_{k-1} = {}_{l+k-2}C_{k-2} + {}_{l+k-2}C_{k-1} = {}_{l+k-1}C_{k-1}.$$

The binomial coefficient ${}_{l+k-1}C_{k-1}$ satisfies the recurrence relation and the initial conditions. Therefore, it is the solution to this recurrence relation and, by Theorem 3, $S(k, l) = {}_{l+k-1}C_{k-1}$. □

This answer, an order of ${}_{l+k-1}C_{k-1}$ for y_n, represents the largest order sufficient to express y_n as a recurrence relation. It is not the least order necessary. The reason for the discrepancy is that the various values for the products of powers of roots might not be distinguishable arithmetically, while in the above proof the various terms were distinguished symbolically. As an example, suppose $x_n = 1^n + 2^n + 3^n + 6^n$ with characteristic equation

$$p(x) = (x-1)(x-2)(x-3)(x-6).$$

The process above indicates that a characteristic polynomial for $y_n = x_n^2$ would be

$$q(x) = (x-1)(x-2)(x-3)(x-6)(x-4)(x-6)(x-12)(x-9)(x-18)(x-36);$$

However, we do not require a double root of 6, obtained on the one hand by $r_1r_4 = 1 * 6 = 6$ and on the other hand by $r_2r_3 = 2 * 3 = 6$.

We can obtain a sharp result if we assume that all the elements of $\alpha \in B_l$ gives rise to a unique value for R^α. We would wish for some general criteria for determining whether all such values are distinct, without arithmetically checking all the possibilities. One such criterion would be the assumption that each root is an integer and each root is divisible by a different prime. Then, given an arithmetic value for a product of roots, we could identify the factors by determining the power of the corresponding prime unique to each root. This would determine the power of the root that comprises the overall product.

4. GENERATING A RELATIONSHIP

One starts with the order k of the recurrence relation for x_n and one decides upon the power l in $y_n = x_n^l$. Next, construct $q(x)$ symbolically, and expand the expression algebraically to obtain the coefficients for $q(x)$ as explicit symmetric polynomials. Finally, write these coefficients in terms of the elementary symmetric polynomials (see Cox [2, pp. 307-09] for more details). The algorithm amounts to successively subtracting appropriate powers of the elementary symmetric polynomials. The powers are obtained by identifying the leading term of the symmetric expression, and using the powers of this monomial to determine powers for products of the elementary symmetric polynomials.

Example: Let $x_n = A_1x_{n-1} + A_2x_{n-2}$, which gives rise to the characteristic polynomial

$$p(x) = (x - r_1)(x - r_2).$$

Notice that $A_1 = r_1 + r_2$ and $A_2 = -r_1r_2$. Let $y_n = x_n^2$, which has derived characteristic polynomial

$$q(x) = (x - r_1^2)(x - r_1r_2)(x - r_2^2).$$

This expands to

$$q(x) = x^3 - (r_1^2 + r_1r_2 + r_2^2)x^2 + (r_1^3r_2 + r_1^2r_2^2 + r_1r_2^3)x - r_1^3r_2^3,$$

which leads to the recurrence relation

$$y_n = (r_1^2 + r_1r_2 + r_2^2)y_{n-1} - (r_1^3r_2 + r_1^2r_2^2 + r_1r_2^3)y_{n-2} + r_1^3r_2^3y_{n-3}.$$

Performing the algorithm indicated in Cox, Little, and O'Shea [2] gives

$$(r_1^2 + r_1r_2 + r_2^2) = (r_1 + r_2)^2 - r_1r_2 = A_1^2 + A_2,$$
$$(r_1^3r_2 + r_1^2r_2^2 + r_1r_2^3) = (r_1 + r_2)^2(r_1r_2) - r_1^2r_2^2 = -A_1^2A_2 - A_2^2,$$
$$r_1^3r_2^3 = (r_1r_2)^3 = -A_2^3.$$

Consequently,

$$y_n = (A_1^2 + A_2)y_{n-1} + (A_1^2A_2 + A_2^2)y_{n-2} - (A_2^3)y_{n-3}. \quad \square$$

5. RESULTS

Using the computer algebra system Maple® and the easier algorithm indicated in [2, pp. 309-10] yielded the following results: If $x_n = ax_{n-1} + bx_{n-2} + cx_{n-3}$ and $y_n = x_n^3$, then

$$y_n = \sum_{i=1}^{10} a_i y_{n-i}$$

with

$$
\begin{aligned}
a_1 &= a^3 + 2ba + c,\\
a_2 &= 2b^3 + ba^4 + 3b^2a^2 + ca^3 + 2cba,\\
a_3 &= 3c^3 + ca^6 - 2b^4a - b^3a^3 + 11c^2ba + 7cba^4 + 5c^2a^3 + 10cb^2a^2,\\
a_4 &= -3cb^3a^3 + 4cb^4a + 2c^3a^3 + 2c^2b^3 - cb^2a^5 + c^2a^6 - 13c^3ba + c^2ba^4 - 13c^2b^2a^2 - 3c^4 - b^6,\\
a_5 &= c^2b^2a^5 - 4c^4ba - 7c^3ba^4 - 5c^4a^3 + 5c^3b^3 - c^3a^6 - cb^6 + 7c^2b^4a - cb^5a^2 + 8c^2b^3a^3,\\
a_6 &= -2c^5a^3 - c^4a^6 + c^2b^6 - 2c^4b^3 - 4c^4ba^4 - 13c^4b^2a^2 - c^3b^4a + c^2b^5a^2 - 3c^3b^3a^3 - 3c^6 - 13c^5ba,\\
a_7 &= -c^4b^3a^3 + 2c^5ba^4 + c^3b^6 - 7c^4b^4a + 10c^5b^2a^2 - 5c^5b^3 + 11c^6ba + 3c^7,\\
a_8 &= -2c^7a^3 + 2c^7ba - b^3c^6 - c^5b^4a + 3a^2c^6b^2,\\
a_9 &= 2c^8ba + c^9 - b^3c^7, \quad \text{and}\\
a_{10} &= -c^{10}.
\end{aligned}
$$

If $x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} + dx_{n-4}$ and $y_n = x_n^2$, then

$$y_n = \sum_{i=1}^{10} a_i y_{n-i}$$

with

$$\begin{aligned}
a_1 &= a^2 + b, \\
a_2 &= b^2 + ba^2 + ca + d, \\
a_3 &= -b^3 + ca^3 + 2c^2 + 4cba, \\
a_4 &= db^2 + 4dba^2 + 5dca + 2d^2 + da^4 - c^2b - cb^2a + c^2a^2, \\
a_5 &= dca^3 - db^2a^2 - 2d^2b + d^2a^2 - dc^2 - 2db^3 - c^3a + c^2b^2, \\
a_6 &= -c^4 + dcb^2a - 2d^3 - 5d^2ca - d^2ba^2 + 4dc^2b - dc^2a^2 - d^2b^2, \\
a_7 &= 4d^2cba - dac^3 - 2a^2d^3 - d^2b^3, \\
a_8 &= -d^3b^2 - cd^3a - d^4 + c^2d^2b, \\
a_9 &= bd^4 - d^3c^2, \quad \text{and} \\
a_{10} &= d^5.
\end{aligned}$$

6. FURTHER RESEARCH

One of the assumptions throughout this article is that the roots $\{r_i \mid i = 1, \ldots, k\}$ are all distinct. The next step in investigating this problem would be to allow for several repeated roots. Each of the roots r_i could have a multiplicity k_i, which would lead to a polynomial of degree $(k_i - 1)$ in n as the coefficient of r_i^n in the Binet form. These different polynomials would then combine in the multinomial theorem to form repeated roots of high degree. Tracing this argument through carefully might yield precise estimates for the order of the recurrence relation $y_n = x_n^l$.

REFERENCES

1. Curtis Cooper & Robert E. Kennedy. "Proof of a Result by Jarden by Generalizing a Proof by Carlitz." *The Fibonacci Quarterly* **33.4** (1995):304-10.
2. David Cox, John Little, and Donald O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and commutative Algebra.* New York: Springer-Verlag, 1992.
3. Thomas Hungerford. *Algebra.* New York: Springer-Verlag, 1974.

AMS Classification Numbers: 11B37, 11C08

❖❖❖