ADVANCED PROBLEMS AND SOLUTIONS

Edited by **Raymond E. Whitney**

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-547 ***Proposed by T. V. Padmakumar, Thycaud, India***

Prove: If p is a prime number, then

$$\left[\sum_{n=1}^{p}\frac{1}{(2n-1)}\right]^2-\left[\sum_{n=1}^{p}\frac{1}{(2n-1)^2}\right]\equiv 0 \pmod{p}.$$

H-548 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequence of Pell numbers by $P_0=0$, $P_1=1$, and $P_{n+2}=2P_{n+1}+P_n$ for $n\ge 0$. Show that if q is a prime such that $q\equiv 1 \pmod 8$ then

$$q\,|\,P_{(q-1)/4} \text{ if and only if } 2^{(q-1)/4}\equiv(-1)^{(q-1)/8} \pmod{q}.$$

H-549 ***Proposed by Paul S. Bruckman, Highwood, IL***

Evaluate the expression:

$$\sum_{n\ge 1}(-1)^{n-1}\tan^{-1}(1/F_{2n}). \tag{1}$$

SOLUTIONS

Exactly Right

H-532 ***Proposed by Paul S. Bruckman, Highwood, IL (Vol. 35, no. 4, November 1997)***

Let $V_n=V_n(x)$ denote the generalized Lucas polynomials defined as follows: $V_0=2$; $V_1=x$; $V_{n+2}=xV_{n+1}+V_n$, $n=0,1,2,\ldots$. If n is an odd positive integer and y is any real number, find all (exact) solutions of the equation: $V_n(x)=y$.

Solution by H.-J. Seiffert, Berlin, Germany

It is well known that $V_n(x)$ is a polynomial of degree n and that, for all complex numbers x, $V_n(x)=\alpha(x)^n+\beta(x)^n$, where

$$\alpha(x)=(x+\sqrt{x^2+4})/2 \quad\text{and}\quad \beta(x)=(x-\sqrt{x^2+4})/2.$$

Here, $\sqrt{x^2+4}$ can be any of the at most two possible roots of x^2+4.

Let $n \in N$ be odd and $y \in R$. We show that the solutions of the equation $V_n(x) = y$ are

$$x = x_k = \sqrt[n]{\alpha(y)}\exp\left(\frac{2k\pi i}{n}\right) + \sqrt[n]{\beta(y)}\exp\left(-\frac{2k\pi i}{n}\right)$$
$$= \left(\sqrt[n]{\alpha(y)} + \sqrt[n]{\beta(y)}\right)\cos\left(\frac{2k\pi}{n}\right) + i\left(\sqrt[n]{\alpha(y)} - \sqrt[n]{\beta(y)}\right)\sin\left(\frac{2k\pi}{n}\right),\quad k = 0, \ldots, n-1.$$

Here, we consider the main branch of the n^{th} root.

Since $\alpha(y) + \beta(y) = y$ is real and n is odd, it is easily seen that $x_0, \ldots, x_{n-1}$ are n distinct complex numbers. However, the equation $V_n(x) = y$ cannot have more than n distinct solutions, so that we are done if we prove that $V_n(x_k) = y$ for $k = 0, \ldots, n-1$.

Since n is odd and $\alpha(y)\beta(y) = -1$, we find

$$x_k^2 + 4 = \sqrt[n]{\alpha(y)^2}\exp\left(\frac{4k\pi i}{n}\right) + \sqrt[n]{\beta(y)^2}\exp\left(-\frac{4k\pi i}{n}\right) + 2,$$

which implies

$$\sqrt{x_k^2 + 4} = \pm\left(\sqrt[n]{\alpha(y)}\exp\left(\frac{2k\pi i}{n}\right) - \sqrt[n]{\beta(y)}\exp\left(-\frac{2k\pi i}{n}\right)\right).$$

It follows that

$$x_k \pm \sqrt{x_k^2 + 4} = 2\sqrt[n]{\alpha(y)}\exp\left(\frac{2k\pi i}{n}\right) \quad\text{and}\quad x_k \mp \sqrt{x_k^2 + 4} = 2\sqrt[n]{\beta(y)}\exp\left(-\frac{2k\pi i}{n}\right).$$

In each case, we have $V_n(x_k) = \alpha(x_k)^n + \beta(x_k)^n = \alpha(y) + \beta(y) = y$.

Also solved by G. Smith and the proposer.

Enter at Your Own Risk

<u>H-533</u> ***Proposed by Andrej Dujella, University of Zagreb, Croatia***
(Vol. 35, no. 4, November 1997)

Let $Z(n)$ be the entry point for positive integers n. Prove that $Z(n) \le 2n$ for any positive integer n. Find all positive integers n such that $Z(n) = 2n$.

Solution by Paul S. Bruckman, Highwood, IL

We first assume that $\gcd(n, 10) = 1$. The following results are well known for all primes $p \ne 2, 5$: $Z(p) \mid (p - (5/p))$; also, $Z(p^e) = p^{e-t}Z(p)$ for some t with $1 \le t \le e$. Then $Z(p^e) = p^{e-t}(p - (5/p))/a$ for some integer $a = a(p)$. If $n = \Pi p^e$, let $n = PQ$, where P consists of those prime powers p^e exactly dividing n and with $a(p) = 1$, and Q is the corresponding product with $a(p) \ge 2$. Note that

$$Z(P) \le 2\prod_{p^e \| P} p^{e-1}\{(p+1)/2\},$$

since $2 \mid ((p - (5/p))$, while

$$Z(Q) \le \prod_{p^e \| Q} p^{e-1}\{(p+1)/2\};$$

therefore,

$$Z(n)=\text{ LCM}\{Z(p^e):p^e\|n\}\le 2\prod_{p^e\|n}p^{e-1}\{(p+1)/2\}.$$

Then

$$Z(n)/n\le 2\prod_{p|n}\{(p+1)/2p\}\le 4/3,$$

since $(p+1)/2p\le 2/3$ for all $p>2$, with equality iff $p=3$.

If $n=5^e m$, where $\gcd(m,10)=1$, then

$$Z(n)=\text{LCM}(Z(5^e),Z(m))=\text{LCM}(5^e,Z(m))\le 5^e\cdot(4m/3)=4n/3.$$

Therefore, $Z(n)\le 4n/3$ for all odd n.

If $n=2m$, where m is odd, then

$$Z(n)=\text{LCM}(3,Z(m))\le 3Z(m)\le 3(4m/3)=4m=2n.$$

If $n=4m$, where m is odd, then

$$Z(n)=\text{LCM}(6,Z(m))\le 6Z(m)\le 6(4m/3)=8m=2n.$$

If $n=2^e m$, where $e\ge 3$ and $\gcd(m,10)=1$, then

$$Z(n)=\text{LCM}(Z(2^e),Z(m))\le\text{LCM}(3\cdot 2^{e-2},Z(m))\le 3\cdot 2^{e-2}\cdot 4m/3=n.$$

In all cases, $Z(n)\le 2n$ for all $n\ge 1$. □

If we examine the various parts of the foregoing proof, we see that $Z(n)$ has a chance of being exactly equal to $2n$ only if 2^1 or 2^2 is the highest power of 2 exactly dividing n. Moreover, if $\gcd(m,30)=1$, if $m>1$, and if $n=2m$ or $4m$, we see that $Z(m)<4m/3$; in this case, $Z(n)<2n$. Note that the factor 5^f of n does not affect the ratio $Z(n)/n$, since $Z(5^f)=5^f$.

Thus, any n with $Z(n)=2n$ must be of the form $2^d\cdot 3^e\cdot 5^f$, where $d=1$ or 2, $e\ge 1$, $f\ge 0$. We observe that $Z(2\cdot 3^e\cdot 5^f)=\text{LCM}(3,4\cdot 3^{e-1},5^f)=12\cdot 5^f$ if $e=1$, or $4\cdot 3^{e-1}\cdot 5^f$ if $e\ge 2$. Thus, $Z(n)=2n$ if $e=1$, $Z(n)<2n$ if $e>1$.

Therefore, if $2^1\|n$, $Z(n)=2n$ iff $e=1$, i.e., iff $n=6\cdot 5^f$. On the other hand, we find that if $n=4\cdot 3^e\cdot 5^f$ then $Z(n)=n$ or $n/3<2n$ in either case.

In conclusion, $Z(n)=2n$ iff $n=6\cdot 5^f$, $f=0,1,2,\ldots$. □

Also solved by L. A. G. Dresel and the proposer.

Representation

H-534 ***Proposed by Piero Filipponi, Rome, Italy***
(Vol. 35, no. 4, November 1997)

An interesting question posed to me by Evelyn Hart (Colgate University, Hamilton, NY) led me to pose, in turn, the following two problems to the readers of *The Fibonacci Quarterly*. (Please see the above volume of the *Quarterly* for a complete statement of Problem H-534.)

Problem A: For k a fixed positive integer, let n_k be *any* integer representable as

$$n_k=\sum_{j=1}^{k}v_jF_j, \tag{1}$$

where v_j equals either j or zero.

Problem B: Is it possible to characterize the set of all positive integers k for which kF_k is representable as

$$kF_k = \sum_{j=1}^{k-1} v_j F_j$$

where v_j is as in Problem A?

Solution by Paul S. Bruckman, Highwood, IL

Solution to Problem A: We first make some notational changes, for convenience. Let $\theta_j = jF_j$. The set of positive integers that may be represented as a sum $\sum_{j=1}^{k} \varepsilon_j \theta j$ with $\varepsilon_j = 0$ or 1, $\varepsilon_k = 1$, is denoted by τ_k. Let $\tau = \bigcup_{k=1}^{\infty} \tau k$. If a positive integer n *cannot* be represented as such a sum for any value of k, we write $n \notin \tau$. Also, define $S(0) = 1$.

We note that we have the following generating function:

$$\prod_{j=1}^{\infty}(1+x^{\theta_j}) = \sum_{k=0}^{\infty} S(k)x^k. \tag{1}$$

We use a comparison test to determine the following result:

$$\lim_{k\to\infty} S(k)/f(k) = 0. \tag{2}$$

The comparison is made with the more well-known generating function:

$$\prod_{j=1}^{\infty}(1+x^{j}) = \sum_{k=0}^{\infty} q(k)x^k, \tag{3}$$

where $q(k)$ is the number of decompositions of k into distinct positive integer summands without regard to order; for example, $q(7) = 5$, since $7 = 1+6 = 2+5 = 3+4 = 1+2+4$. Since the θ_j's are natural numbers, it is clear that

$$0 \le S(k) \le q(k), \;\; k = 0, 1, 2, \ldots. \tag{4}$$

Indeed, all of the $q(k)$'s are > 0. The following asymptotic formula (paraphrased to conform with our notation) is given in [1]:

$$q(k) \sim \frac{1}{4}(3k^3)^{-1/4} \exp(\pi\sqrt{k/3}), \text{ as } k \to \infty. \tag{5}$$

Thus, $\log q(k) \sim \pi\sqrt{k/3}$. On the other hand, $\log f(k) \sim k \log \alpha$. Hence, $\log\{q(k)/f(k)\} \to -\infty$, which implies $\lim_{k\to\infty} q(k)/f(k) = 0$. This, together with (4), implies (3). □

Partial Solution to Problem B: We see that if $\theta_k = \sum_{j=1}^{k-1} \varepsilon_j \theta_j$ then $\varepsilon_{k-1} = 1$ and $k \ge 7$, by the proposer's comments. For, otherwise, $\theta_k \le f(k-2) = \theta_k - L_{k+1} + 2$, which is clearly impossible. Therefore, either $\theta_k \in \tau_{k-1}$ or $\theta_k \notin \tau$. For brevity, we let U denote the set of $k \ge 7$ such that $\theta_k \in \tau_{k-1}$. Note that $S(\theta_k) \ge 1$ for all $k \ge 1$. One way to characterize U, albeit not a very satisfactory way from a theoretical standpoint, is to observe that U is precisely the set of k such that $S(\theta_k) \ge 2$; this, however, is little more than a restatement of the definition of the $S(k)$'s.

Some other observations may be made, which may or may not be useful. For example, we can determine the characteristic polynomial of the θ_k's. The following relation is easily found:

$$\theta_k - \theta_{k-1} - \theta_{k-2} = L_{k-1}. \tag{6}$$

Thus, the characteristic, or "annihilating," polynomial of the θ_k's is $(z^2-z-1)^2 = z^4-2z^3-z^2+2z+1$; that is, we have the pure recurrence

$$\theta_k - 2\theta_{k-1} - \theta_{k-2} + 2\theta_{k-3} + \theta_{k-4} = 0. \tag{7}$$

We may define the following quantity:

$$u_k \equiv 2\theta_k + \theta_{k-1},\ k = 1, 2, \dots \text{ (with } \theta_0 \equiv 0\text{)}. \tag{8}$$

Then we may recast (7) as follows:

$$u_{k-1} - u_{k-3} = \theta_k,\ k = 4, 5, \dots. \tag{9}$$

A consequence of these relations is the following:

$$u_{2k} + 2 = \sum_{i=0}^{k} \theta_{2i+1},\quad u_{2k-1} = \sum_{i=1}^{k} \theta_{2i},\quad k = 1, 2, \dots. \tag{10}$$

This shows that u_{2k-1} and $(u_{2k}+2)$ are elements of τ_{2k} and τ_{2k+1}, respectively. We also see that

$$u_k + u_{k-1} + 2 = f(k+1),\ k = 2, 3, \dots. \tag{11}$$

It is not clear at this point how these relations may be useful in determining which values of k are "acceptable," in the sense that $k \in U$. We observe from (6), however, that if $L_{k-1} \in \tau_m$ for some $m \le k-3$ then $k \in U$.

One practical approach is simply to expand the generating function to any desired number of terms and pick out the values of k for which $S(k) \ge 2$. To ensure that we are not omitting some values of k that *eventually* generate $S(k) \ge 2$, we need to take enough terms in the product. If the partial products $\prod_{j=1}^{n}(1+x^{\theta_j})$ have the expansion $\sum_{k=0}^{f(n)} S(k,n)x^k$, and if the integer $\mu = \mu(k)$ is determined from $\theta_\mu \le k < \theta_{\mu+1}$ then $S(k,n) = S(k)$ for all $n \ge \mu$. In particular, $S(\theta_k, n) = S(\theta_k)$ for all $n \ge \theta_k$.

We conclude with a table indicating the first 25 values of θ_k, $S(k)$, and $f(k)$, also indicating all acceptable representations of θ_k as an element of τ_{k-1} for $k \ge 7$, if such representations exist. We denote such representations in an abbreviated form, where the indicated m-tuple gives the subscripts r of the θ_r's entering in the representation, shown in descending order.

The table was *not* generated by expansion, as might be suggested by the previous comments. Rather, we used a constructive algorithm for generating the representations (if any) in τ_{k-1} of θ_k. Following is a brief description of the algorithm.

We begin by assuming that $\theta_k \in \tau_{k-1}$ and compute the difference $N_1 \equiv \theta_k - \theta_{k-1}$. There exists an index r such that $\theta_r \le N_1 < \theta_{r+1}$. The next term is either θ_r or θ_{r-1}. If $N_1 > f(r-1)$, such next term *must* be θ_r. If $N_1 \le f(r-1)$, such next term is either θ_r or θ_{r-1}; both cases are possible *a priori* and must be examined separately. Let $N_2 = N_1 - \theta_s$, where θ_s is the next term selected (i.e., $s = r$ or $r-1$) and repeat the process with N_2. The algorithm continues until a final difference N_ω, say, is either determined to be representable as a sum of the θ_j's or recognized as impossible to be thus represented. Note: If $N_j = f(m)$ for some m and j, we may either stop at the term θ_m or replace $f(m)$ by $\theta_1 + \theta_2 + \cdots + \theta_m$. Keeping track of all "forks in the road" (where two choices were possible *a priori*), we thereby generate all possible representations, if any.

It is tempting on the basis of the data, to make the conjecture that $k \in U$ for *all* values except 1, 2, 3, 4, 5, 6, 8, 8, and 14. It would seem unlikely that $S(\theta_k) = 1$ for any value of $k > 25$, but these methods did not resolve this question.

TABLE

k	θ_k	$S(k)$	τ_{k-1} Representation(s)	$f(k)$
1	1	1	–	1
2	2	1	–	3
3	6	1	–	9
4	12	1	–	21
5	25	1	–	46
6	48	1	–	94
7	91	2	{6,5,4,3}	185
8	168	1	–	353
9	306	1	–	659
10	550	2	{9,8,6,5,2,1}	1209
11	979	2	{10,9,7,5,3,1}	2188
12	1728	2	{11,10,8,5,3}	3916
13	3029	2	{12,11,8,7,6,4,2,1}	6945
14	5278	1	–	12223
15	9150	3	{14,13,10,8,7,5,3,2,1}, {14,12,11,10,9,8,7,6,2}	21373
16	15792	3	{15,14,11,9,6,5,3}, {15,13,12,11,10,9,6,2}	37165
17	27149	3	{16,15,12,9,7,6,5,3,2,1}, {16,14,13,12,11,9,5,4}	64314
18	46512	4	{17,16,13,9,8,6,4,3,2}, {17,16,12,11,10,8,7,6,3,1}, {17,15,14,13,12,7,6,5,4,2}	110826
19	79439	5	{18,17,14,9,8,5,1}, {18,17,13,12,10,9,7,6,5,1}, {18,16,15,14,12,11}, {18,16,15,14,12,10,9,7,5,3,1}	190265
20	135300	5	{19,18,15,8,5,3}, {19,18,14,6,4,2,1}, {19,18,14,13,10,9,8,4,3}, {19,17,16,15,12,11,10,9,8,5,4,2}	325565
21	229866	4	{20,19,15,13,12,11,8,6,5}, {20,18,17,16,13,12,9,6,2}, {20,18,17,16,13,11,10,9,8,6,5,3,2}	555431
22	389642	2	{21,19,18,16,15,14,13,10,5,1}	945073
23	659111	4	{22,21,17,15,12,11,9,8,7,5,3,1}, {22,20,19,17,16,15,12,10,9,6,3,1}, {22,20,19,17,16,14;13,12,11,10,8,6,3,2,1}	1604184
24	1112832	8	{23,22,18,15,14,13,7,4,3,1}, {23,22,17,16,15,14,13,12,11,10,9,7,5,2}, {23,21,20,19,14,13,10,8,6,5,4,3}, {23,21,20,18,17,15,14,9,7,6,4,3,2,1}, {23,21,20,19,14,13,10,8,7}, {23,21,20,1118,17,15,13,12,10,9,7,5,4,2,1}	2717016
25	1875625	8	{24,23,19,16,14,12,11,9,7,6,4,3,2,1}, {24,23,18,17,16,15,13,12,8,7,6,4,2,1}, {24,22,21,20,14,12,10,9,7,5,3,1}, {24,22,21,20,14,12,11}, {24,22,21,19,18,16,11,10,4,1}, {24,22,21,19,18,15,14,12,11,8,5,3}, {24,22,21,19,18,15,14,12,10,9,8,7,6,4,2,1}	4592641

Reference

1. M. Abramowitz & I. A. Stegun, eds. *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards. Ninth printing, Nov. 1970 (with corrections), p. 826.

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