ON THE OCCURRENCE OF F_n IN THE ZECKENDORF DECOMPOSITION OF nF_n

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1. INTRODUCTION

The Zeckendorf decomposition of a natural number n is the unique expression of n as a sum of Fibonacci numbers with nonconsecutive indices and with each index greater than 1, where $F_0 = 0$, $F_1 = 1$, and $F_{i+1} = F_i + F_{i-1}$ form the Fibonacci numbers for $i \geq 0$ (see [6] and [8], or see pages 108-09 in [7]). The Zeckendorf decomposition of products of the form kF_m for $k, m \in \mathbb{N}$ is studied in [2] and [5]. For each positive integer n, let *Ratio*(n) be the ratio of the number of $k \in \mathbb{N}$ with $k \leq n$ that do have F_k in the Zeckendorf decomposition of kF_k to those that do not. In this paper we prove Conjecture 1 from [2], which essentially states that as $n \to \infty$ we have *Ratio*$(n) \to \beta^{-2}$, where $\beta = (1+\sqrt{5})/2$. This result, Theorem 4.9, is proved using methods introduced in [5] by Hart.

The β-expansion of a natural number n, first introduced in [1], is the unique finite sum of integral, nonconsecutive powers of β that equals n. Grabner et al., in [3] and [4], prove that for $m \geq \log_\beta k$ the Zeckendorf decomposition of kF_m can be produced by replacing each β^i in the β-expansion of k with F_{m+i}. Thus, our result also answers a question posed by Bergman in [1] that asks for the frequency of the occurrence of β^0 in the β-expansions of the natural numbers. For simplicity, all results and proofs in the rest of the paper will be stated in terms of β-expansions.

Our proof entails first finding formulas for *Ratio*(L_{2k}) and *Ratio*(L_{2k+1}), where $L_0 = 2$, $L_1 = 1$, and $L_{i+1} = L_i + L_{i-1}$ form the Lucas sequence for $i \geq 0$. We prove that as $k \to \infty$ the two sequences of ratios for odd- and even-indexed Lucas numbers both decrease to β^{-2}. We then prove that for values of n between two Lucas numbers we have *Ratio*(n) trapped between the two sequences.

The recursive pattern we have discovered in the β-expansions, and upon which our proof is based, can be used to find the frequency of the occurrence of other powers of β as well. This extension of the current problem will be addressed in a future paper.

2. DEFINITIONS AND PRELIMINARIES

We use definitions and notation similar to those in [5]. In particular, $\ell(n)$ denotes the absolute value of the smallest power of β in the β-expansion of n, and $u(n)$ denotes the largest such power.

The following is a restatement of Theorem 1 from [4] in terms of the β-expansion.

Theorem 2.1 (Grabner et al.): For $k \geq 1$, we have $\ell(n) = u(n) = 2k$ whenever $L_{2k} \leq n \leq L_{2k+1}$, and we have $\ell(n) = 2k+2$ and $u(n) = 2k+1$ whenever $L_{2k+1} < n < L_{2k+2}$.

Definition 2.2: We define V to be the infinite dimensional vector space over $\mathbb{Z}$ given by $V := \{(\ldots, v_{-1}, \underline{v_0}, v_1, v_2, \ldots) : v_i \in \mathbb{Z} \forall i$, with at most finitely many v_i nonzero$\}$. For convenience, we underline the zeroth coordinate.

Definition 2.3: Define $\hat{V}$ to be the subset of V consisting of all vectors whose entries are in the set $\{0, 1\}$ and which have no two consecutive ones. We will call the elements of $\hat{V}$ totally reduced vectors.

As in [5], we represent β-expansions by vectors of ones and zeros, where a one in the j^{th} coordinate represents β^j. The powers of β increase from left to right in the vector.

Definition 2.4: We define the function $\beta : \mathbb{N} \to \hat{V}$ so that, when the β-expansion of n is $\sum_{i=-\infty}^{\infty} e_i \beta^i$, $\beta(n)$ is the vector in $\hat{V}$ with $v_i = e_i$.

Definition 2.5: The function $\sigma : V \to \mathbb{N}$ is defined as follows: $\sigma((\ldots, v_{-1}, v_0, v_1, \ldots)) = \sum_{i=-\infty}^{\infty} v_i \beta^i$.

Thus, $\sigma(\beta(n)) = n$ for all natural numbers n. (Note that the definition of σ in [5] is in terms of Fibonacci numbers and is not equivalent to the one given here. Specifically, the two functions are only guaranteed to be equal when applied to $\beta(n)$ where $n \in \mathbb{N}$.)

Figure 1 shows the vector representations of the β-expansion of the first 30 natural numbers. Note that the coefficient of β^0 is underlined.

n									$\beta(n)$								
1									$\underline{1}$								
2							1	0	$\underline{0}$	1							
3							1	0	$\underline{0}$	0	1						
4							1	0	$\underline{1}$	0	1						$4 = L_3$
5					1	0	0	1	$\underline{0}$	0	0	1					
6					1	0	0	0	$\underline{0}$	1	0	1					
7					1	0	0	0	$\underline{0}$	0	0	0	1				$7 = L_4$
8					1	0	0	0	$\underline{1}$	0	0	0	1				
9					1	0	1	0	$\underline{0}$	1	0	0	1				
10					1	0	1	0	$\underline{0}$	0	1	0	1				
11					1	0	1	0	$\underline{1}$	0	1	0	1				$11 = L_5$
12			1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1			
13			1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1			
14			1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1			
15			1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1			
16			1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1			
17			1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1			
18			1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1		$18 = L_6$
19			1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1		
20			1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1		
21			1	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1		
22			1	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	1		$22 = 2L_5$
23			1	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	1		
24			1	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	1		
25			1	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	1		$25 = L_6 + L_4$
26			1	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	1		
27			1	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	1		
28			1	0	1	0	1	0	$\underline{0}$	0	1	0	1	0	1		
29			1	0	1	0	1	0	$\underline{1}$	0	1	0	1	0	1		$29 = L_7$
30	1	0	0	1	0	1	0	1	$\underline{0}$	0	0	0	0	0	0	1	

FIGURE 1. β-Expansions

Following [2], we say that n has property $\mathcal{P}$ if β^0 appears in the β-expansion of n.

Definition 2.6: For natural numbers n, m, we let

$$Ones(n,m) = |\{k \in \mathbb{N} : n < k \le m,\ k \text{ has property } \mathcal{P}\}|,$$
$$Zeros(n,m) = |\{k \in \mathbb{N} : n < k \le m,\ k \text{ does not have property } \mathcal{P}\}|.$$

We also define, for $n > 1$, $Ratio(n)$ to be $Ones(0,n)/Zeros(0,n)$.

We will be using the following known facts about Fibonacci and Lucas numbers:

$$\lim_{x\to\infty}(F_x / F_{x-1}) = \beta; \tag{1}$$

$$\lim_{x\to\infty}(F_x / F_{x-2}) = 1+\beta = \beta^2; \tag{2}$$

$$F_{n+h}F_{n+k} - F_n F_{n+h+k} = (-1)^n F_h F_k. \tag{3}$$

Formula (3) is from [7], page 177, (20a).

Note that in [3] and [4] the indices for Fibonacci and Lucas numbers are different from the indices used here. We use $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$.

3. THE RATIO FOR LUCAS NUMBERS

Our first goal is to prove the following proposition.

Proposition 3.1: For $k \ge 1$,

$$Ratio(L_{2k}) = \frac{F_{2k-1}}{F_{2k+1}} \quad \text{and} \quad Ratio(L_{2k+1}) = \frac{F_{2k}+1}{F_{2k+2}-1}.$$

Thus, both ratios decrease to β^{-2} as k increases.

We shall devote this section to developing the facts needed for the proof of Proposition 3.1. Recall that we express β-expansions of integers with powers of β increasing from left to right.

Lemma 3.2:

(1) $\beta(L_{2k}) = (10^{2k-1}\underline{0}0^{2k-1}1)$ for $k \ge 1$.

(2) $\beta(2L_{2k-1}) = (1000(10)^{k-2}\underline{1}(01)^{k-2}0001)$ for $k \ge 2$.

(3) $\beta(L_{2k} + L_{2k-2}) = (1010^{2k-3}\underline{0}0^{2k-3}101)$ for $k \ge 2$.

(4) $\beta(L_{2k+1}) = ((10)^k\underline{1}(01)^k)$ for $k \ge 1$.

(5) $\beta(2L_{2k}) = (10010^{2k-2}\underline{0}0^{2k-3}1001)$ for $k \ge 2$.

(6) $\beta(L_{2k+1} + L_{2k-1}) = (100100(10)^{k-2}\underline{1}(01)^{k-1}001)$ for $k \ge 2$.

Proof: Parts (1), (2), and (4) follow from results in [2] and from the relationship between the Zeckendorf expansion of nF_n and the β-expansion of n as developed in [3] and [4] (see the Introduction to this paper). Part (3) follows from part (1) when we apply Proposition 4.4 from [5]. Parts (5) and (6) are proved in [5]. □

Lemma 3.3:

(1) $\beta(L_{2k}-1)=(10^{2k-1}\underline{0}(10)^{k-1}1)$ for $k\geq 1$.

(2) $\beta(L_{2k+1}+1)=(10(01)^k\underline{0}0^{2k}1)$ for $k\geq 1$.

(3) $\beta(2L_{2k}-1)=(10010^{2k-2}\underline{0}(10)^{k-1}001)$ for $k\geq 2$.

(4) $\beta(L_{2k+1}+L_{2k-1}+1)=(1000(01)^{k-1}\underline{0}0^{2k-2}101)$ for $k\geq 2$.

Proof: We make repeated applications of Theorem 5.8 from [5]. If $v\in V$, $w=\mathscr{A}(v)$ is the vector in $\hat{V}$ obtained by applying the algorithm in [5]; by the properties of this algorithm, $\sigma(v)=\sigma(w)$.

Part (1): Define $v(k)=(10^{2k-1}\underline{0}(10)^{k-1}1)$ and $w(k)=(\underline{0}(10)^{k-1}1)$. We show that $\mathscr{A}(w(k)+(\underline{1}))=(\underline{0}0^{2k-1}1)$ by induction on k. For $k=1$, $\mathscr{A}(w(1)+(\underline{1}))=\mathscr{A}((\underline{1}1))=(\underline{0}01)$. Assume that $k\geq 2$ and that the formula is correct for smaller k. Then

$$\begin{aligned}\mathscr{A}(w(k)+(\underline{1}))&=\mathscr{A}(w(k-1)+(\underline{0}0^{2k-2}1)+(\underline{1}))=\mathscr{A}(\mathscr{A}(w(k-1)+(\underline{1}))+(\underline{0}0^{2k-2}1))\\&=\mathscr{A}((\underline{0}0^{2k-3}1)+(\underline{0}0^{2k-2}1))=\mathscr{A}((\underline{0}0^{2k-3}11))=(\underline{0}0^{2k-1}1).\end{aligned}$$

Since $v(k)=w(k)+(10^{2k-1}\underline{0})$, we have

$$\begin{aligned}\mathscr{A}(v(k)+(\underline{1}))&=\mathscr{A}(\mathscr{A}(w(k)+(\underline{1}))+(10^{2k-1}\underline{0}))\\&=\mathscr{A}((\underline{0}0^{2k-1}1)+(10^{2k-1}\underline{0}))=(10^{2k-1}\underline{0}0^{2k-1}1)=\beta(L_{2k}).\end{aligned}$$

By uniqueness of the β-representation, this implies that $\beta(L_{2k}-1)=v(k)$.

Part (2): This is proved by induction on k. For $k=1$, we have

$$\beta(L_3+1)=\mathscr{A}((10\underline{1}01)+(\underline{1}))=\mathscr{A}((10\underline{2}01))=\mathscr{A}((20\underline{0}11))=(1001\underline{0}001).$$

For the inductive step, we assume that $k\geq 2$. We have

$$\begin{aligned}\beta(L_{2k+1}+1)&=\beta(L_{2k-1}+L_{2k}+1)=\mathscr{A}(\beta(L_{2k-1}+1)+\beta(L_{2k}))\\&=\mathscr{A}((10(01)^{k-1}\underline{0}0^{2k-2}1)+(10^{2k-1}\underline{0}0^{2k-1}1))=\mathscr{A}((20(01)^{k-1}\underline{0}0^{2k-2}11))\\&=\mathscr{A}((1001(01)^{k-1}\underline{0}0^{2k-2}001))=(10(01)^k\underline{0}0^{2k}1).\end{aligned}$$

Part (3): Using part (1), we have

$$\begin{aligned}\beta(2L_{2k}-1)&=\beta(L_{2k}-1+L_{2k})=\mathscr{A}(\beta(L_{2k}-1)+\beta(L_{2k}))\\&=\mathscr{A}((10^{2k-1}\underline{0}(10)^{k-1}1)+(10^{2k-1}\underline{0}0^{2k-1}1))=\mathscr{A}((20^{2k-1}\underline{0}(10)^{k-1}11))\\&=(10010^{2k-2}\underline{0}(10)^{k-1}001).\end{aligned}$$

Part (4): We use induction on k. For $k=2$, we have

$$\begin{aligned}\beta(L_5+L_3+1)&=\mathscr{A}(\beta(L_5+1)+\beta(L_3))=\mathscr{A}((100101\underline{0}00001)+(10\underline{1}01))\\&=\mathscr{A}((100111\underline{1}01001))=\mathscr{A}((100110\underline{0}11001))=(100001\underline{0}00101).\end{aligned}$$

For the inductive step, we assume that $k\geq 3$. We have

$$\begin{aligned}\beta(L_{2k+1}+L_{2k-1}+1) &= \beta(L_{2k-1}+L_{2k}+L_{2k-3}+L_{2k-2}+1)\\ &= \mathscr{A}(\beta(L_{2k-1}+L_{2k-3}+1)+\beta(L_{2k}+L_{2k-2}))\\ &= \mathscr{A}((1000(01)^{k-2}\underline{0}0^{2k-4}101)+(1010^{2k-3}\underline{0}0^{2k-3}101))\\ &= \mathscr{A}((2010(01)^{k-2}\underline{0}0^{2k-4}1111)).\end{aligned}$$

We apply the algorithm and have

$$\beta(L_{2k+1}+L_{2k-1}+1)=\mathscr{A}((100110(01)^{k-2}\underline{0}0^{2k-4}00101))=(1000(01)^{k-1}\underline{0}0^{2k-2}101). \quad \square$$

Note that in Figure 1, for $L_4<n\le L_5$, $\beta(n)$ and $\beta(L_5+n)$ are identical in the coordinate positions within 3 positions of the center. The same relationship can be observed for $L_3<n\le L_4$ between $\beta(n)$ and $\beta(L_6+n)$. These types of relationships are described in Lemma 3.8. We now define a transformation that will allow us to discuss these relationships in a precise way.

We define $\mathscr{S}(v,T)$ to be the vector obtained by switching (from 0 to 1 or vice versa) the values of all entries of v whose coordinates are in a finite set $T\subset\mathbb{Z}$. This is equivalent to adding and subtracting powers of β. [In our applications, when applying the transformation to $\beta(n)$ we will switch only those entries with coordinate positions close to $u(n)$ or $-\ell(n)$, and will leave the central entries unchanged.]

Definition 3.4: Let $v\in\hat{V}$. Let T be a finite set of coordinates. Define $w=\mathscr{S}(v,T)\in V$ to be the vector with $w_i\in\{0,1\}$ for all $i\in\mathbb{Z}$ and with $w_i\ne v_i$ if $i\in T$ and $w_i=v_i$ if $i\notin T$.

Definition 3.5: If $v,w\in\hat{V}$ and T is a finite set of coordinates, then we say $v\equiv_T w$ (v is congruent to w mod T) if $v_i=w_i$ $\forall i\in T$.

Lemma 3.6: Let $n,m,x\in\mathbb{N}$ and let T be a finite set of coordinates. Suppose that $\mathscr{S}(\beta(n),T)=\beta(m)$, $\beta(n)\equiv_T\beta(n+x)$ and $\mathscr{S}(\beta(n+x),T)\in\hat{V}$. Then $\mathscr{S}(\beta(n+x),T)=\beta(m+x)$.

Proof: We are given β-expansions for n and x as follows: $n=\sum_{i=-\infty}^{\infty}\varepsilon_i\beta^i$ and $x=\sum_{i=-\infty}^{\infty}\delta_i\beta^i$. Let $T(0)=T\cap\{i:\varepsilon_i=0\}$ and let $T(1)=T\cap\{i:\varepsilon_i=1\}$.

Let $d=\sum_{i\in T(0)}\beta^i-\sum_{i\in T(1)}\beta^i$. The fact that $\mathscr{S}(\beta(n),T)=\beta(m)$ means that $n+d=m$. Because $n,m\in\mathbb{N}$, we have $d\in\mathbb{Z}$.

We know that $\mathscr{S}(\beta(n+x),T)\in\hat{V}$, which means that $\mathscr{S}(\beta(n+x),T)$ is the β-expansion of some real number. Since $\beta(n)\equiv_T\beta(n+x)$, we have $\sigma(\mathscr{S}(\beta(n+x),T))=n+x+d=m+x\in\mathbb{N}$. The β-expansion of a natural number is unique, so $\mathscr{S}(\beta(n+x),T)=\beta(m+x)$. $\square$

Corollary 3.7: Let n_1,n_2,m_1,m_2 be natural numbers where $n_1\le n_2, m_1\le m_2$. Let T be a finite set of coordinates such that $0\notin T$ and, for $0\le x\le n_2-n_1$, $\beta(n_1+x)\equiv_T\beta(n_1)$, $\mathscr{S}(\beta(n_1+x),T)\in\hat{V}$, $\mathscr{S}(\beta(n_1),T)=\beta(m_1)$, and $\mathscr{S}(\beta(n_2),T)=\beta(m_2)$. Then $\mathscr{S}(\beta(n_1+x),T)=\beta(m_1+x)$ and, thus, $Ones(n_1,n_1+x)=Ones(m_1,m_1+x)$ and $Zeros(n_1,n_1+x)=Zeros(m_1,m_1+x)$.

Figure 1 above and Figure 2 below illustrate the relationships in parts (6) and (10) of Lemma 3.8. The formulas from Lemmas 3.2 and 3.3 are used in the proof.

n									$\beta(n)$									
1									$\underline{1}$									
2							1	0	$\underline{0}$	1								
3							1	0	$\underline{0}$	0	1							
4							1	0	$\underline{1}$	0	1							
5					1	0	0	1	$\underline{0}$	0	0	1						
6					1	0	0	0	$\underline{0}$	1	0	1						
7					1	0	0	0	$\underline{0}$	0	0	0	1					$7 = L_4$
8					1	0	0	0	$\underline{1}$	0	0	0	1					
9					1	0	1	0	$\underline{0}$	1	0	0	1					
10					1	0	1	0	$\underline{0}$	0	1	0	1					
11					1	0	1	0	$\underline{1}$	0	1	0	1					$11 = L_5$
12			1	0	0	1	0	1	$\underline{0}$	0	0	0	0	1				
13			1	0	0	1	0	0	$\underline{0}$	1	0	0	0	1				
14			1	0	0	1	0	0	$\underline{0}$	0	1	0	0	1				
15			1	0	0	1	0	0	$\underline{1}$	0	1	0	0	1				
16			1	0	0	0	0	1	$\underline{0}$	0	0	1	0	1				
17			1	0	0	0	0	0	$\underline{0}$	1	0	1	0	1				
18			1	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	1			$18 = L_6$
19			1	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	1			
20			1	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	1			
21			1	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	1			
22			1	0	0	0	1	0	$\underline{1}$	0	1	0	0	0	1			
23			1	0	1	0	0	1	$\underline{0}$	0	0	1	0	0	1			
24			1	0	1	0	0	0	$\underline{0}$	1	0	1	0	0	1			
25			1	0	1	0	0	0	$\underline{0}$	0	0	0	1	0	1			
26			1	0	1	0	0	0	$\underline{1}$	0	0	0	1	0	1			
27			1	0	1	0	1	0	$\underline{0}$	1	0	0	1	0	1			
28			1	0	1	0	1	0	$\underline{0}$	0	1	0	1	0	1			
29			1	0	1	0	1	0	$\underline{1}$	0	1	0	1	0	1			$29 = L_7$
30	1	0	0	1	0	1	0	1	$\underline{0}$	0	0	0	0	0	0	1		
31	1	0	0	1	0	1	0	0	$\underline{0}$	1	0	0	0	0	0	1		
32	1	0	0	1	0	1	0	0	$\underline{0}$	0	1	0	0	0	0	1		
33	1	0	0	1	0	1	0	0	$\underline{1}$	0	1	0	0	0	0	1		
34	1	0	0	1	0	0	0	1	$\underline{0}$	0	0	1	0	0	0	1		
35	1	0	0	1	0	0	0	0	$\underline{0}$	1	0	1	0	0	0	1		
36	1	0	0	1	0	0	0	0	$\underline{0}$	0	0	0	1	0	0	1		$36 = 2L_6$
37	1	0	0	1	0	0	0	0	$\underline{1}$	0	0	0	1	0	0	1		
38	1	0	0	1	0	0	1	0	$\underline{0}$	1	0	0	1	0	0	1		
39	1	0	0	1	0	0	1	0	$\underline{0}$	0	1	0	1	0	0	1		
40	1	0	0	1	0	0	1	0	$\underline{1}$	0	1	0	1	0	0	1		$40 = L_7 + L_5$
41	1	0	0	0	0	1	0	1	$\underline{0}$	0	0	0	0	1	0	1		
42	1	0	0	0	0	1	0	0	$\underline{0}$	1	0	0	0	1	0	1		
43	1	0	0	0	0	1	0	0	$\underline{0}$	0	1	0	0	1	0	1		
44	1	0	0	0	0	1	0	0	$\underline{1}$	0	1	0	0	1	0	1		
45	1	0	0	0	0	0	0	1	$\underline{0}$	0	0	1	0	1	0	1		
46	1	0	0	0	0	0	0	0	$\underline{0}$	1	0	1	0	1	0	1		
47	1	0	0	0	0	0	0	0	$\underline{0}$	0	0	0	0	0	0	0	1	$47 = L_8$
48	1	0	0	0	0	0	0	0	$\underline{1}$	0	0	0	0	0	0	0	1	
49	1	0	0	0	0	0	1	0	$\underline{0}$	1	0	0	0	0	0	0	1	
50	1	0	0	0	0	0	1	0	$\underline{0}$	0	1	0	0	0	0	0	1	

FIGURE 2. More β-Expansions

Lemma 3.8: For $k \geq 2$ with *Digits* = *Ones* throughout or *Digits* = *Zeros* throughout:

(1) If $0 \leq x \leq L_{2k-3}$, then

$$\begin{aligned} Digits(L_{2k-2}, L_{2k-2}+x) &= Digits(L_{2k}, L_{2k}+x) \\ &= Digits(L_{2k}+L_{2k-2}, L_{2k}+L_{2k-2}+x) \\ &= Digits(2L_{2k}, 2L_{2k}+x). \end{aligned}$$

(2) If $0 \le x \le L_{2k-2}$, then
$$\begin{aligned} Digits(L_{2k-1}, L_{2k-1}+x) &= Digits(2L_{2k+1}, 2L_{2k+1}+x) \\ &= Digits(L_{2k+1}, L_{2k+1}+x) \\ &= Digits(L_{2k+1}+L_{2k-1}, L_{2k+1}+L_{2k-1}+x). \end{aligned}$$

(3) $Digits(L_{2k-2}, L_{2k-1}) = Digits(L_{2k}, 2L_{2k-1})$.

(4) $Digits(L_{2k-3}, L_{2k-2}) = Digits(2L_{2k-1}, L_{2k}+L_{2k-2})$.

(5) $Digits(L_{2k-2}, L_{2k-1}) = Digits(L_{2k}+L_{2k-2}, L_{2k+1})$.

(6) $Digits(L_{2k}, L_{2k+1}) = 2Digits(L_{2k-2}, L_{2k-1}) + Digits(L_{2k-3}, L_{2k-2})$.

(7) $Digits(L_{2k-1}, L_{2k}) = Digits(L_{2k+1}, 2L_{2k})$.

(8) $Digits(L_{2k-2}, L_{2k-1}) = Digits(2L_{2k}, L_{2k+1}+L_{2k-1})$.

(9) $Digits(L_{2k-1}, L_{2k}) = Digits(L_{2k+1}+L_{2k-1}, L_{2k+2})$.

(10) $Digits(L_{2k+1}, L_{2k+2}) = 2Digits(L_{2k-1}, l_{2k}) + Digits(L_{2k-2}, L_{2k-1})$.

(11) For $L_{2k+1} < n \le L_{2k+2}$, $\beta(n)$ starts with 100 [i.e., the values at coordinate positions $-\ell(n)$, $-\ell(n)+1$, and $-\ell(n)+2$ of $\beta(n)$ are 1, 0, and 0, respectively].

Proof: Let $T_1 = \{-2k, -2k+2, 2k-2, 2k\}$. It can be checked that $\mathcal{S}(\beta(L_{2k-2}), T_1) = \beta(L_{2k})$ and that $\mathcal{S}(\beta(L_{2k-1}), T_1) = \beta(2L_{2k-1})$. It follows from 2.1 that, for all n with $L_{2k-2} \le n \le L_{2k-1}$, $\beta(n) \equiv_T \beta(L_{2k-2})$ and $\mathcal{S}(\beta(n), T_1) \in \hat{V}$. Let $x = n - L_{2k-2}$ so that $0 \le x \le L_{2k-3}$ and note that, by 3.7, $Digits(L_{2k-2}, L_{2k-2}+x) = Digits(L_{2k}, L_{2k}+x)$.

Let $T_2 = \{-2k-2, 2k+2\}$. It can be checked that $\mathcal{S}(\beta(L_{2k-1}), T_2) = \beta(2L_{2k+1})$ and that $\mathcal{S}(\beta(L_{2k}), T_2) = \beta(L_{2k+2}+L_{2k})$. By 2.1, for all n satisfying $L_{2k-1} \le n \le L_{2k}$, $\beta(n) \equiv_{T_2} \beta(L_{2k-1})$ and $\mathcal{S}(\beta(n), T_2) \in \hat{V}$. Let $x = n - L_{2k-1}$ so that $0 \le x \le L_{2k-2}$ and note that, by 3.7, $Digits(L_{2k-1}, L_{2k-1}+x) = Digits(2L_{2k+1}, 2L_{2k+1}+x)$.

Let $T_3 = \{-2k, 2k\}$. Then $\mathcal{S}(\beta(L_{2k-2}), T_3) = \beta(L_{2k}+L_{2k-2})$, $\mathcal{S}(\beta(L_{2k-1}), T_3) = \beta(L_{2k+1})$ and, for all n satisfying $L_{2k-2} \le n \le L_{2k-1}$, $\beta(n) \equiv_{T_3} \beta(L_{2k-2})$, and $\mathcal{S}(\beta(n), T_3) \in \hat{V}$. Let $x = n - L_{2k-2}$ so that $0 \le x \le L_{2k-3}$ and note that, by 3.7, $Digits(L_{2k-2}, L_{2k-2}+x) = Digits(L_{2k}+L_{2k-2}, L_{2k} + L_{2k-2}+x)$.

Using the largest values of x possible in the above arguments, we have formulas (3) and (5) of the lemma proven as well as formula (4) for $k \ge 3$. To complete the proof of formula (4), we check by hand that it holds for $k = 2$ as well. Thus,
$$\begin{aligned} &Digits(L_{2k}, L_{2k+1}) \\ &= Digits(L_{2k}, 2L_{2k-1}) + Digits(2L_{2k-1}, L_{2k}+L_{2k-2}) + Digits(L_{2k}+L_{2k-2}, L_{2k+1}) \\ &= Digits(L_{2k-2}, L_{2k-1}) + Digits(L_{2k-3}, L_{2k-2}) + Digits(L_{2k-2}, L_{2k-1}), \end{aligned}$$
which proves (6).

Formulas (7)-(11) remain to be proven as well as the third equality from (1) and the second and third equalities from (2). The proof for these remaining formulas is by induction on k. For $k = 2$, the formulas can be checked directly. Assume $k \ge 3$ and all the remaining formulas hold for smaller k.

Let $T_4=\{-2k-2,-2k,-2k+1,2k-1,2k+1\}$. Note that $\mathcal{S}(\beta(L_{2k-1}+1),T_4)=\beta(L_{2k+1}+1)$ and $\mathcal{S}(\beta(L_{2k}-1),T_4)=\beta(2L_{2k}-1)$. By 2.1 and part (11) of the induction hypothesis, we see that for all n where $L_{2k-1}<n<L_{2k}$, $\beta(n)\equiv_{T_4}\beta(L_{2k-1}+1)$ and $\mathcal{S}(\beta(n),T_4)\in\hat{V}$. Let $x=n-L_{2k-1}$ so that $0<x\le L_{2k-2}-1$. We have $Digits(L_{2k-1}+1,L_{2k-1}+x)=Digits(L_{2k+1}+1,L_{2k+1}+x)$. Since $(\beta(L_{2k-1}+1))_0=(\beta(L_{2k+1}+1))_0$, we have $Digits(L_{2k-1},L_{2k-1}+x)=Digits(L_{2k+1},L_{2k+1}+x)$ for $0<x\le L_{2k-2}-1$. When $x=0$ the equality is trivially true. We note that $(\beta(L_{2k}))_0=(\beta(2L_{2k}))_0$ so, for $x=L_{2k-2}$, we have $Digits(L_{2k-1},L_{2k})=Digits(L_{2k+1},2L_{2k})$. Note also that the above shows that, for $L_{2k-1}<n\le L_{2k}$, $\mathcal{S}(\beta(n),T_4)$ starts with 100 and, hence, $\beta(\ell)$ starts with 100 for $L_{2k+1}<\ell\le 2L_{2k}$. We have proven (7), some of (11), and the second equality of (2).

Let $T_5=\{-2k-2,-2k+1,-2k+2,2k+1\}$. We can check that $\mathcal{S}(\beta(L_{2k-2}),T_5)=\beta(2L_{2k})$, $\mathcal{S}(\beta(L_{2k-1}),T_5)=\beta(L_{2k+1}+L_{2k-1})$ and, for all n satisfying $L_{2k-2}\le n\le L_{2k-1}$, $\beta(n)\equiv_{T_5}\beta(L_{2k-2})$ and $\mathcal{S}(\beta(n),T_5)\in\hat{V}$. Let $x=n-L_{2k-2}$ so that $0\le x\le L_{2k-3}$ and note that, by 3.7, we have $Digits(L_{2k-2},L_{2k-2}+x)=Digits(2L_{2k},2L_{2k}+x)$. Therefore, $Digits(L_{2k-2},L_{2k-1})=Digits(2L_{2k},$ $L_{2k+1}+L_{2k-1})$. Using 2.1, we have, for $L_{2k-2}\le n\le L_{2k-1}$, that $\mathcal{S}(\beta(n),T_5)$ starts with 100; thus, $\beta(\ell)$ starts with 100 for $2L_{2k}\le\ell\le L_{2k+1}+L_{2k-1}$. We have proven (8), some more of (11), and the last equality from (1).

Let $T_6=\{-2k-2,-2k,2k+1\}$. Then we have $\mathcal{S}(\beta(L_{2k-1}+1),T_6)=\beta(L_{2k+1}+L_{2k-1}+1)$ and $\mathcal{S}(\beta(L_{2k}-1),T_6)=\beta(L_{2k+2}-1)$. For all n satisfying $L_{2k-1}<n<L_{2k}$, we have $\beta(n)\equiv_{T_6}\beta(L_{2k-1}+1)$ and $\mathcal{S}(\beta(n),T_6)\in\hat{V}$. Let $x=n-L_{2k-1}$ so that $0<x\le L_{2k-2}-1$. Then $Digits(L_{2k-1}+1,L_{2k-1}+x)=$ $Digits(L_{2k+1}+L_{2k-1}+1,L_{2k+1}+L_{2k-1}+x)$. We note that $(\beta(L_{2k-1}+1))_0=(\beta(L_{2k+1}+L_{2k-1}+1))_0$, so we actually have $Digits(L_{2k-1},L_{2k-1}+x)=Digits(L_{2k+1}+L_{2k-1},L_{2k+1}+L_{2k-1}+x)$ for $0\le x\le$ $L_{2k-2}-1$. Since also $(\beta(L_{2k}))_0=(\beta(L_{2k+2}))_0$ we have, for $x=L_{2k-2}$, that $Digits(L_{2k-1},L_{2k})=$ $Digits(L_{2k+1}+L_{2k-1},L_{2k+2})$. We have proven (9) and the last equality in (2).

The above also shows that, for $\ell_{2k-1}<n<L_{2k}$, we know that $\mathcal{S}(\beta(n),T_6)$ starts with 100. Thus, for $L_{2k+1}+L_{2k-1}+1\le\ell\le L_{2k+2}-1$, $\beta(\ell)$ starts with 100. After we check that $\beta(\ell)$ starts with 100 for $\ell=L_{2k+2}$, we have completed the proof of (11).

Hence,

$$\begin{aligned}Digits(L_{2k+1},L_{2k+2})&=Digits(L_{2k+1},2L_{2k})+Digits(2L_{2k},L_{2k+1}+L_{2k-1})\\&\quad+Digits(L_{2k+1}+L_{2k-1},L_{2k+2})\\&=2Digits(L_{2k-1},L_{2k})+Digits(L_{2k-2},L_{2k-1}),\end{aligned}$$

and (10) is proven. □

Proposition 3.9: For $k\ge1$:

(1) $Ones(L_{2k},L_{2k+1})=F_{2k-2}+1$ and $Zeros(L_{2k},L_{2k+1})=F_{2k}-1$.

(2) $Ones(L_{2k+1},L_{2k+2})=F_{2k-1}-1$ and $Zeros(L_{2k+1},L_{2k+2})=F_{2k+1}+1$.

(3) $Ones(0,L_{2k})=F_{2k-1}$ and $Zeros(0,L_{2k})=F_{2k+1}$.

(4) $Ones(0,L_{2k+1})=F_{2k}+1$ and $Zeros(0,L_{2k+1})=F_{2k+2}-1$.

Proof: The first two results are proved by induction on k. It may be checked by inspection that the formulas hold for $k=1$ and $k=2$. Let $k\geq 3$ and assume that the formulas hold for smaller k. Then, by 3.8,

$$Ones(L_{2k}, L_{2k+1}) = 2Ones(L_{2k-2}, L_{2k-1}) + Ones(L_{2k-3}, L_{2k-2}) = 2F_{2k-4} + 2 + F_{2k-5} - 1 = F_{2k-2} + 1$$

by the induction hypothesis. Similarly, using Lemma 3.8 and the induction hypothesis

$$Ones(L_{2k+1}, L_{2k+2}) + Ones(L_{2k-2}, L_{2k-1}) = 2F_{2k-3} - 2 + F_{2k-4} + 1 = F_{2k-1} - 1.$$

To prove the last two results, we note that

$$Ones(L_{2k}, L_{2k+2}) = Ones(L_{2k}, L_{2k+1}) + Ones(L_{2k+1}, L_{2k+2}) = F_{2k}.$$

So

$$\begin{aligned} Ones(0, L_{2k}) &= 1 + \sum_{i=1}^{k-1} Ones(L_{2i}, L_{2i+2}) = 1 + \sum_{i=1}^{k-1} F_{2i} \\ &= F_1 + F_2 + \cdots + F_{2k-2} = F_{2k-1}. \end{aligned}$$

And

$$Ones(0, L_{2k+1}) = Ones(0, L_{2k}) + Ones(L_{2k}, L_{2k+1}) = F_{2k-1} + F_{2k-2} + 1 = F_{2k} + 1. \quad \square$$

Proof of 3.1: The formulas for $Ratio(L_{2k})$ and $Ratio(L_{2k+1})$ follow from 3.9. The limit follows from equation (2). To see that both sequences are decreasing, we use equation (3). We have $(F_{2k+1})^2 - F_{2k-1}F_{2k+3} = -1 < 0$, which implies that $Ratio(L_{2k}) > Ratio(L_{2k+2})$. We also have $(F_{2k})^2 - F_{2k-2}F_{2k+2} = 1 < F_{2k+2} - F_{2k-2}$. This implies that $Ratio(L_{2k+1}) < Ratio(L_{2k-1})$. $\square$

4. THE RATIO FOR NON-LUCAS NUMBERS

In this section we prove that the sequence $Ratio(n)$ for $n \geq 2$ is trapped between the two decreasing sequences of Proposition 3.1, which both approach β^{-2}.

Proposition 4.1: Let $k\geq 1$. Then, for $n \in \mathbb{N}$ with $L_{2k} < n \leq L_{2k+2}$, $Ratio(L_{2k}) \leq Ratio(n) \leq Ratio(L_{2k-1})$.

We devote the rest of the paper to developing the lemmas which, when combined with some of the results of the previous section, will allow us to prove Proposition 4.1.

The following lemma will be used repeatedly.

Lemma 4.2: Let $a,b,c,d \in \mathbb{N}$ and $x, y \in \mathbb{R}$. If $\frac{a}{b} \leq x$ and $\frac{c}{d} \leq y$, then $\frac{a+c}{b+d} \leq \max\{x,y\}$. When each $\leq$ is replaced by $\geq$, the result holds with *max* replaced by *min*.

Lemma 4.3: For $k \geq 1$:

(1) $Ones(0, 2L_{2k-1}) = 2F_{2k-2} + 1$, $Zeros(0, 2L_{2k-1}) = 2F_{2k} - 1$.

(2) $Ones(0, L_{2k} + L_{2k-2}) = F_{2k-1} + F_{2k-3}$, $Zeros(0, L_{2k} + L_{2k-2}) = F_{2k+1} + F_{2k-1}$.

(3) $Ones(0, 2L_{2k}) = 2F_{2k-1}$, $Zeros(0, 2L_{2k}) = 2F_{2k+1}$.

(4) $Ones(0, L_{2k+1} + L_{2k-1}) = F_{2k} + F_{2k-2} + 1$, $Zeros(0, L_{2k+1} + L_{2k-1}) = F_{2k+2} + F_{2k} - 1$.

Proof: We use Lemma 3.8 and Proposition 3.9. For example, for $k \geq 2$, $Ones(0, 2L_{2k-1}) = Ones(0, L_{2k}) + Ones(L_{2k}, 2L_{2k-1}) = Ones(0, L_{2k}) + Ones(L_{2k-2}, L_{2k-1})$, using part (1) of Lemma

3.8. Therefore, $Ones(0, 2L_{2k-1}) = F_{2k-1} + F_{2k-4} + 1 = 2F_{2k-2} + 1$. The case $k = 1$ can be checked directly. The rest of the proofs are similar. □

Lemma 4.4 $\{F_{2k-2} / F_{2k}\}_k$ is an increasing sequence that approaches β^{-2} as $k \to \infty$.

Proof: Apply equations (2) and (3). □

Lemma 4.5: Let $k \geq 1$. If $0 < x < L_{2k+2}$, then $Ones(L_{2k+1}, L_{2k+1} + x) / Zeros(L_{2k+1}, L_{2k+1} + x) \leq \beta^{-2}$.

Proof: In the proof of this lemma, *Digits* stands for either *Ones* or *Zeros*. The proof is by induction on *k*. The cases for $k = 1$ and $k = 2$ can be checked directly. Assume $k \geq 3$, and the result is true for smaller *k*.

Case 1. $1 \leq x \leq L_{2k-2}$. By Lemma 3.8 and the induction hypothesis,

$$\frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} = \frac{Ones(L_{2k-1}, L_{2k-1} + x)}{Zeros(L_{2k-1}, L_{2k-1} + x)} \leq \beta^{-2}.$$

Case 2. $L_{2k-2} < x < L_{2k-1}$. Let $y = x - L_{2k-2}$ and $z = y + L_{2k-4}$. Then $0 < y < L_{2k-3}$ and $L_{2k-4} < z < L_{2k-2}$. This implies that $L_{2k+1} + x = 2L_{2k} + y$. We have

$$\frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} = \frac{Ones(L_{2k+1}, 2L_{2k}) + Ones(2L_{2k}, 2L_{2k} + y)}{Zeros(L_{2k+1}, 2L_{2k}) + Zeros(2L_{2k}, 2L_{2k} + y)}.$$

Note that using Lemma 3.8,

$$\begin{aligned} Digits(2L_{2k}, 2L_{2k} + y) &= Digits(L_{2k-2}, L_{2k-2} + y) \\ &= Digits(L_{2k-3}, L_{2k-2} + y) - Digits(L_{2k-3}, L_{2k-2}) \\ &= Digits(L_{2k-3}, L_{2k-3} + z) - Digits(L_{2k-3}, L_{2k-2}). \end{aligned}$$

Hence, using Lemma 4.3 and Proposition 3.9, we see that

$$\begin{aligned} \frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} &= \frac{2F_{2k-1} - (F_{2k} + 1) - F_{2k-3} + (F_{2k-4} + 1) + Ones(L_{2k-3}, L_{2k-3} + z)}{2F_{2k+1} - (F_{2k+2} - 1) - F_{2k-1} + (F_{2k-2} - 1) + Zeros(L_{2k-3}, L_{2k-3} + z)} \\ &= \frac{F_{2k-4} + Ones(L_{2k-3}, L_{2k-3} + z)}{F_{2k-2} + Zeros(L_{2k-3}, L_{2k-3} + z)} \leq \beta^{-2}, \end{aligned}$$

since $F_{2k-4} / F_{2k-2} \leq \beta^{-2}$ by Lemma 4.4 and $Ones(L_{2k-3}, L_{2k-3} + z) / Zeros(L_{2k-3}, L_{2k-3} + x) \leq \beta^{-2}$ by the induction hypothesis.

Case 3. $x = L_{2k-1}$. Then

$$\frac{Ones(L_{2k+1}, L_{2k+1} + L_{2k-1})}{Zeros(L_{2k+1}, L_{2k+1} + L_{2k-1})} = \frac{F_{2k} + F_{2k-2} + 1 - (F_{2k} + 1)}{F_{2k+2} + F_{2k} - 1 - (F_{2k+2} - 1)} = \frac{F_{2k-2}}{F_{2k}} \leq \beta^{-2}$$

using Proposition 3.9, Lemma 4.3, and Lemma 4.4.

Case 4. $L_{2k-1} < x \leq L_{2k}$. Let $y = x - L_{2k-1}$. So $0 < y < L_{2k-2}$. We have

$$\frac{Ones(L_{2k+1}, L_{2k+1} + x)}{Zeros(L_{2k+1}, L_{2k+1} + x)} = \frac{Ones(L_{2k+1}, L_{2k+1} + L_{2k-1}) + Ones(L_{2k-1}, L_{2k-1} + y)}{Zeros(L_{2k+1}, L_{2k+1} + L_{2k-1}) + Zeros(L_{2k-1}, L_{2k-1} + y)}$$

$$= \frac{F_{2k} + F_{2k-2} + 1 - (F_{2k}+1) + Ones(L_{2k-1}, L_{2k-1}+y)}{F_{2k+2} + F_{2k} - 1 - (F_{2k+2}-1) + Zeros(L_{2k-1}, L_{2k-1}+y)}$$

$$= \frac{F_{2k-2} + Ones(L_{2k-1}, L_{2k-1}+y)}{F_{2k} + Zeros(L_{2k-1}, L_{2k-1}+y)} \le \beta^{-2}$$

using Lemma 3.8, Proposition 3.9, Lemma 4.3, Lemma 4.4, and the induction hypothesis.

Case 5. $L_{2k} < x < L_{2k+1}$. Let $y = x - L_{2k}$ and $z = y + L_{2k-2}$. Then $0 < y < L_{2k-1}$ and $L_{2k-2} < z < L_{2k}$. As before, we have

$$\begin{aligned} &Digits(L_{2k+1}, L_{2k+1}+x) = Digits(L_{2k+1}, L_{2k+2}+y) \\ &= Digits(L_{2k+1}, L_{2k+2}) + Digits(L_{2k}, L_{2k}+y) \\ &= Digits(L_{2k+1}, L_{2k+2}) + Digits(L_{2k-1}, L_{2k-1}+z) - Digits(L_{2k-1}, L_{2k}). \end{aligned}$$

Thus,

$$\frac{Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(L_{2k+1}, L_{2k+1}+x)} = \frac{F_{2k-1} - 1 - (F_{2k-3}-1) + Ones(L_{2k-1}, L_{2k-1}+z)}{F_{2k+1} + 1 - (F_{2k-1}+1) + Zeros(L_{2k-1}, L_{2k-1}+z)}$$

$$= \frac{F_{2k-2} + Ones(L_{2k-1}, L_{2k-1}+z)}{F_{2k} + Zeros(L_{2k-1}, L_{2k-1}+z)} \le \beta^{-2}$$

using the induction hypothesis.

Case 6. $x = L_{2k+1}$. We have

$$\frac{Ones(L_{2k+1}, 2L_{2k+1})}{Zeros(L_{2k+1}, 2L_{2k+1})} = \frac{(2F_{2k}+1) - (F_{2k}+1)}{(2F_{2k+2}-1) - (F_{2k+2}-1)}$$

$$= \frac{F_{2k}}{F_{2k+2}} \le \beta^{-2}.$$

Case 7. $L_{2k+1} < x \le 2L_{2k}$. Let $y = x - L_{2k+1}$. So $0 < y \le L_{2k-2}$. We have, by the induction hypothesis and using Lemma 4.3,

$$\frac{Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(L_{2k+1}, L_{2k+1}+x)} = \frac{Ones(L_{2k+1}, 2L_{2k+1}) + Ones(L_{2k-1}, L_{2k-1}+y)}{Zeros(L_{2k+1}, 2L_{2k+1}) + Zeros(L_{2k-1}, L_{2k-1}+y)}$$

$$= \frac{(2F_{2k}+1) - (F_{2k}+1) + Ones(L_{2k-1}, L_{2k-1}+y)}{(2F_{2k+2}-1) - (F_{2k+2}-1) + Zeros(L_{2k-1}, L_{2k-1}+y)}$$

$$= \frac{F_{2k} + Ones(L_{2k-1}, L_{2k-1}+y)}{F_{2k+2} + Zeros(L_{2k-1}, L_{2k-1}+y)} \le \beta^{-2}.$$

Case 8. $2L_{2k} < x < L_{2k+2}$. Let $y = x - 2L_{2k}$ and let $z = y + L_{2k-2}$. Then $0 < y < L_{2k-1}$ and $L_{2k-2} < z < L_{2k}$. We have

$$\begin{aligned} &Digits(L_{2k+1}, L_{2k+1}+x) = Digits(L_{2k+1}, L_{2k+2}+L_{2k}) + Digits(L_{2k+2}+L_{2k}, L_{2k+2}+L_{2k}+y) \\ &= Digits(L_{2k+1}, L_{2k+2}+L_{2k}) + Digits(L_{2k}, L_{2k}+y) \\ &= Digits(L_{2k+1}, L_{2k+2}+L_{2k}) + Digits(L_{2k-1}, L_{2k-1}+z) - Digits(L_{2k-1}, L_{2k}). \end{aligned}$$

Thus,

$$\frac{Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(L_{2k+1}, L_{2k+1}+x)} = \frac{F_{2k+1}+F_{2k-1}-(F_{2k}+1)-(F_{2k-3}-1)+Ones(L_{2k-1}, L_{2k-1}+z)}{F_{2k+3}+F_{2k+1}-(F_{2k+2}-1)-(F_{2k-1}+1)+Zeros(L_{2k-1}, L_{2k-1}+z)}$$

$$= \frac{F_{2k}+Ones(L_{2k-1}, L_{2k-1}+z)}{F_{2k+2}+Zeros(L_{2k-1}, L_{2k-1}+z)} \leq \beta^{-2}$$

by the induction hypothesis. □

Proposition 4.6: Let $k \geq 1$. If $0 < x < L_{2k+2}$, then $Ratio(L_{2k+1}+x) \leq Ratio(L_{2k+1})$.

Proof:

$$Ratio(L_{2k+1}+x) = \frac{Ones(0, L_{2k+1})+Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(0, L_{2k+1})+Zeros(L_{2k+1}, L_{2k+1}+x)} \leq Ratio(L_{2k+1})$$

since

$$\frac{Ones(L_{2k+1}, L_{2k+1}+x)}{Zeros(L_{2k+1}, L_{2k+1}+x)} \leq \beta^{-2} \leq Ratio(L_{2k+1})$$

by Lemma 4.5 and Proposition 3.1. □

Lemma 4.7: Let $k \geq 1$. If $0 < x < L_{2k+1}$, then

$$\frac{Ones(L_{2k}, L_{2k}+x)}{Zeros(L_{2k}, L_{2k}+x)} \geq Ratio(L_{2k}).$$

Proof: The proof of this lemma is similar to that of Lemma 4.5 and is omitted.

Proposition 4.8: Let $k \geq 1$. If $0 < x < L_{2k+1}$, then $Ratio(L_{2k}+x) \geq Ratio(L_{2k})$.

Proof: For $x > 1$, we have

$$Ratio(L_{2k}+x) = \frac{Ones(0, L_{2k})+Ones(L_{2k}, L_{2k}+x)}{Zeros(0, L_{2k})+Zeros(L_{2k}, L_{2k}+x)} \geq Ratio(L_{2k})$$

by Proposition 4.7 and Lemma 4.2. For $x = 1$, $Ones(L_{2k}, L_{2k}+x) = 1$, $Zeros(L_{2k}, L_{2k}+x) = 0$, and the result follows. □

Proof of Proposition 4.1: By Propositions 4.6 and 4.8, it follows for $k \geq 1$ that, if $L_{2k+1} \leq n \leq L_{2k+2}$, then $Ratio(L_{2k}) \leq Ratio(n) \leq Ratio(L_{2k+1}) \leq Ratio(L_{2k-1})$. Also, for $k \geq 1$, if $L_{2k} < n < L_{2k+1}$, then $Ratio(L_{2k}) \leq Ratio(n) \leq Ratio(L_{2k-1})$. □

The following theorem has now been proven.

Theorem 4.9: The limit of the ratio of natural numbers having Property $\mathcal{P}$ to those not having Property $\mathcal{P}$ is $\lim_{n\to\infty} Ratio(n) = \beta^{-2}$.

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CORRIGENDUM

In the November 1998 issue of *The Fibonacci Quarterly* (Vol. 36, no. 5), Clark Kimberling's article entitled "Edouard Zeckendorf" appeared on pages 416-418. Due to an unfortunate printing error, the signature which was to accompany the article was inadvertently omitted. We apologize to the author, and are pleased to present the missing signature below: