ON A GENERALIZATION OF THE BINOMIAL THEOREM

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1. INTRODUCTION

The elementary binomial theorem is arguably one of the oldest and perhaps most well-known result in mathematics. This famous theorem, which was known to Chinese mathematicians from as early as the thirteenth century, has been subject since that time to a number of generalizations, one of which is attributable to Newton. In this result, commonly referred to today as the General Binomial Theorem, Newton asserted that the expansion of $(1+x)^n$ for negative and fractional exponents consisted of the following series

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots+\frac{n(n-1)\cdots(n-p+1)}{p!}x^p+\cdots, \tag{1}$$

where the variable x was assumed "small." This binomial series was applied to great effect by Newton in such diverse problems as the quadrature of the hyperbola, root extraction, and the approximation of π. In contrast, the second and perhaps more obvious extension to the binomial theorem can be found in the so-called multinomial theorem of Leibniz, where the expansion of a general multinomial

$$(x_1+x_2+\cdots+x_m)^n \tag{2}$$

into a polynomial of m variables was considered (see [1], p. 340). This particular result, which has found numerous applications in the area of combinatorics, is somewhat more "algebraic" in character when compared with the former generalization, which is essentially a statement concerning the power series representation of a function. In keeping with the "algebraic" spirit of (2), we present in this paper an additional extension to the binomial theorem via the development of an expansion theorem for the following class of polynomial functions, denoted

$$(x)_{a_n} = \prod_{i=1}^{n}(x+a_i), \tag{3}$$

in which the sequence $\{a_n\}$ of complex numbers is assumed in arithmetic progression. It should be noted that the construction of this expansion theorem can be viewed as a "connection constant" problem of the Umbral Calculus (see [4], p. 120) in which real numbers c_{nk} are sought so that a given polynomial sequence $p_n(x)$ can be expanded in terms of another, as follows:

$$p_n(x) = \sum_{k>0} c_{nk}q_k(x).$$

In this article we shall not make use of the Umbral Calculus to derive the desired expansion theorem; rather, we shall be content with applying more elementary methods to effect the said result. The outline of this paper is as follows. To facilitate the main result, it will first be necessary to formulate an expression for the coefficients within the polynomial expansion of (3) in terms of the elements of an arbitrary sequence. This is achieved in Section 2, where the coefficient of x^{n-p} for $p=1,2,\ldots,n$, denoted $\phi_p(n)$, will be shown to consist of a p-fold summation of a p-fold product. When $\{a_n\}$ is substituted with an arithmetic progression, these summands then reduce, as demonstrated in Section 3, to a linear combination of binomial coefficients as follows:

$$\phi_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+p+1-m}{2p+1-m}. \tag{4}$$

Moreover, the scalars $\theta_m^{(p)}$, which vary in accordance with the particular arithmetic progression chosen, will be calculated via an accompanying algorithm, thereby determining completely the equation for the coefficient of x^{n-p}. This use of an algorithm in the formulation of $\phi_p(n)$ highlights one major difficulty when attempting to construct a general expansion theorem for (3), namely that, in most instances, no simple closed-form expression exists for $\theta_m^{(p)}$ in terms of the parameters m and p. However, all such apparent difficulties diminish when dealing with a constant sequence (say $a_n = a$), as the corresponding scalars will assume the following simple form,

$$\theta_m^{(p)} = \begin{cases} 0 & \text{for } m = 1, 2, \ldots, p, \\ a^p & \text{for } m = p+1, \end{cases}$$

which, when combined with equations (3) and (4), will yield the binomial theorem. An alternate expansion theorem is also derived when $\{a_n\}$ is in geometric progression. Finally, in Section 4, we will explore an application of the above expansion theorem to the Pochhammer family of polynomial functions that result when $a_n = n-1$. Of particular interest will be the derivation of closed-form expressions for the Stirling numbers of first order, which shall mirror existing formulas for the Stirling numbers of second order (see [6], p. 233).

2. PRELIMINARIES

In this section we shall be concerned with the expansion of a class of polynomial functions which result from the n-fold binomial product $(x+a_1)(x+a_2)\cdots(x+a_n)$ for a given sequence $\{a_n\}$. Our aim is to derive a closed-form expression for the coefficients within these polynomial expansions in terms of the elements of $\{a_n\}$. We begin with a formal definition.

Definition 2.1: Let $\{a_n\}$ be an arbitrary sequence of complex numbers. Then the following n-fold binomial product $(x+a_1)(x+a_2)\cdots(x+a_n)$ shall be denoted by $(x)_{a_n}$. In addition, the coefficient of x^{n-p} for $p = 1, 2, \ldots, n$ within the polynomial expansion of $(x)_{a_n}$ will be written as $\phi_p(n)$.

Remark 2.1: The notation $(x)_{a_n}$ has been improvised from the Pochhammer symbol $(x)_n$, which denotes the rising factorial polynomial of degree n given by $x(x+1)\cdots(x+n-1)$.

It is clear from the definition that each coefficient $\phi_p(n)$ in $(x)_{a_n}$ is an elementary symmetric function in $a_1, a_2, \ldots, a_n$. Although it is well known (see [2], p. 252) that these functions can be expressed in terms of a multiple summation of a p-fold product, the formulation provided is somewhat incomplete for our purposes here. This is the motivation behind the following discussion, which will lead to a more satisfactory representation of $\phi_p(n)$ in Proposition 2.1. We return now to the expansion of $(x)_{a_n}$.

To determine how the coefficients within $(x)_{a_n}$ are formed by the terms of an arbitrary sequence, let us examine $\phi_p(n)$ for $p = 1, 2, 3$ in the cases $n = 2, \ldots, 5$. Beginning with $\phi_1(n)$, it is evident upon expanding that the coefficient of x^{n-1} is equal to the n^{th} partial sum of the sequence $\{a_n\}$. Next, by grouping lower-order terms in each expansion, we observe the following for increasing n:

$$\begin{aligned}
\phi_2(2) &= a_2a_1,\\
\phi_2(3) &= a_2a_1 + a_3(a_1+a_2),\\
\phi_2(4) &= a_2a_1 + a_3(a_1+a_2) + a_4(a_1+a_2+a_3).
\end{aligned}$$

Thus, it would appear, at least empirically, that $\phi_2(n)$ consists of a summation of $n-1$ terms, each of which is the sum of a 2-fold product. Therefore, if the outer and inner terms of each product in the above summands were indexed by i_1 and i_2, respectively, one may then infer that

$$\phi_2(n) = \sum_{i_1=1}^{n-1} a_{i_1+1}\left\{\sum_{i_2=1}^{i_1} a_{i_2}\right\} = \sum_{i_1=1}^{n-1}\sum_{i_2=1}^{i_1} a_{i_1+1}a_{i_2}. \tag{5}$$

Finally, for simplicity, set $\tilde{\phi}_2(n) = \phi_2(n+1)$. Then a similar arrangement of lower-order terms reveals

$$\begin{aligned}
\phi_3(3) &= a_3\tilde{\phi}_2(1),\\
\phi_3(4) &= a_3\tilde{\phi}_2(1) + a_4\tilde{\phi}_2(2),\\
\phi_3(5) &= a_3\tilde{\phi}_2(1) + a_4\tilde{\phi}_2(2) + a_5\tilde{\phi}_2(3).
\end{aligned}$$

Once again we are presented with a clear pattern in which $\phi_3(n)$ appears to consist of a summation of $n-2$ terms each of the form $a_{i_1+2}\tilde{\phi}_2(i_1)$. Thus, after relabeling index variables from i_m to i_{m+1} in (5), we propose

$$\phi_3(n) = \sum_{i_1=1}^{n-2} a_{i_1+2}\{\tilde{\phi}_2(i_1)\} = \sum_{i_1=1}^{n-2}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1+2}a_{i_2+1}a_{i_3}. \tag{6}$$

Consequently, with the aid of equations (5) and (6), one may conjecture that $\phi_p(n)$ is formed from a p-fold summation of the product $a_{i_1+p-1}a_{i_2+p-2}\cdots a_{i_p}$ in which the index of the outer summand assumes the values $i_1 = 1, 2, \ldots, n-p+1$ with all subsequent indexes i_m ranging over $i_m = 1$, $2, \ldots, i_{m-1}$ for $m = 2, 3, \ldots, p$. By continuing as above and using (5) and (6), one may construct similar expressions for the coefficients of lower-order powers in $(x)_{a_n}$ that are in agreement with the previously suggested rule of formation. Hence, we now consider the following result, which is stated in terms of elementary symmetric functions.

Proposition 2.1: Suppose $\{a_n\}$ is an arbitrary sequence, then for $n = 2, 3, \ldots$ the elementary symmetric function $\phi_p(n)$ in $a_1, a_2, \ldots, a_n$ is given by

$$\phi_p(n) = \begin{cases} \displaystyle\sum_{i_1=1}^{n} a_{i_1} & \text{for } p = 1,\\[2ex] \displaystyle\sum_{i_1=1}^{n-p+1}\sum_{i_2=1}^{i_1}\cdots\sum_{i_p=1}^{i_{p-1}} a_{i_1+p-1}a_{i_2+p-2}\cdots a_{i_p} & \text{for } p = 2, 3, \ldots, n. \end{cases} \tag{7}$$

Proof: Fix the sequence $\{a_n\}$ in question and set $n = 2$ as the base for the following inductive argument. Clearly, (7) holds for the case $n = 2$ since

$$\phi_1(2)=\sum_{i_1=1}^{2} a_{i_1}=a_1+a_2 \quad \text{and} \quad \phi_2(2)=\sum_{i_1=1}^{1}\sum_{i_2=1}^{i_1} a_{i_1+1}a_{i_2}=a_1a_2,$$

which are in agreement with the coefficients found in the expansion

$$(x+a_1)(x+a_2)=x^2+(a_1+a_2)x+a_1a_2.$$

Assume the result holds for $n=k$ where $k\geq 2$. Thus,

$$(x)_{a_k}=x^k+\phi_1(k)x^{k-1}+\phi_2(k)x^{k-2}+\cdots+\phi_k(k), \tag{8}$$

where the coefficients $\phi_p(k)$ are of the form as stated above. Multiplying (8) by the term $(x+a_{k+1})$ and collecting like powers of x yields a polynomial of degree $k+1$ with coefficients defined as follows:

$$\phi_1(k+1)=a_{k+1}+\phi_1(k), \tag{9}$$

$$\phi_p(k+1)=a_{k+1}\phi_{p-1}(k)+\phi_p(k) \text{ for } p=2,3,\ldots,k, \tag{10}$$

$$\phi_{k+1}(k+1)=a_{k+1}\phi_k(k). \tag{11}$$

From this set of equations we now generate via the inductive hypothesis corresponding expressions for $\phi_p(k+1)$. Beginning with (9), it is immediately apparent that

$$\phi_1(k+1)=\sum_{i_1=1}^{k+1} a_{i_1}.$$

Now from (10) we have, for $p=2,3,\ldots,k$,

$$\begin{aligned}\phi_p(k+1)=a_{k+1}&\sum_{i_1=1}^{k-p+2}\sum_{i_2=1}^{i_1}\cdots\sum_{i_{p-1}=1}^{i_{p-2}} a_{i_1+p-2}a_{i_2+p-3}\cdots a_{i_{p-1}}\\ &+\sum_{i_1=1}^{k-p+1}\sum_{i_2=1}^{i_1}\cdots\sum_{i_p=1}^{i_{p-1}} a_{i_1+p-1}a_{i_2+p-2}\cdots a_{i_p}.\end{aligned} \tag{12}$$

Relabeling index variables from i_m to i_{m+1} in the expression for $\phi_{p-1}(k)$, observe that $a_{k+1}\phi_{p-1}(k)$ is equal to

$$a_{i_1+p-1}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\cdots\sum_{i_p=1}^{i_{p-1}} a_{i_2+p-2}a_{i_3+p-3}\cdots a_{i_p}$$

when $i_1=k-p+2$. Consequently, by factoring a_{i_1+p-1} in the above $(p-1)$-fold summation and adding the result to the second summand of (12) yields

$$\phi_p(k+1)=\sum_{i_1=1}^{k-p+2}\sum_{i_2=1}^{i_1}\cdots\sum_{i_p=1}^{i_{p-1}} a_{i_1+p-1}a_{i_2+p-2}\cdots a_{i_p}.$$

Finally, from (11), we deduce $\phi_{k+1}(k+1)=a_{k+1}a_k\cdots a_1$ which, clearly, is in agreement with the hypothesized expression for the coefficient of x^0 in $(x)_{a_{k+1}}$. Thus, the result holds for $n=k+1$. Hence, by induction, (7) is valid for all $n=2,3,\ldots$. □

3. MAIN RESULTS

With the formulation in Section 2 of a precise relationship between the coefficients of $(x)_{a_n}$ and the elements of $\{a_n\}$, it is now possible to determine, for suitable classes of sequences, explicit algebraic expressions for $\phi_p(n)$ in terms of the parameters n and p. Clearly, those sequences of interest must possess a closed-form expression for their respective partial sums. However, in general, this will not guarantee the existence of explicit formulas for subsequent $\phi_p(n)$, as the following simple example indicates. Let $a_n = \frac{1}{n(n+1)}$, then an elementary calculation establishes $\phi_1(n) = \frac{n}{n+1}$. This, in turn, implies that

$$\phi_2(n) = \sum_{i_1=1}^{n-1} a_{i_1+1}\left\{\sum_{i_2=1}^{i_1} a_{i_2}\right\} = \sum_{i_1=1}^{n-1} \frac{i_1}{(i_1+1)^2(i_1+2)},$$

which cannot be expressed as a rational function in n due to the presence of the factor $\frac{1}{(i_1+1)^2}$.

Remark 3.1: We note that the function $\phi_2(n)$ in the previous example can be written as the sum of a rational function in n and the di-gamma function $\psi'(z)$. Indeed, by decomposing into partial fractions, observe that

$$\begin{aligned}\phi_2(n) &= \sum_{i_1=1}^{n-1}\left\{\frac{2}{i_1+1} - \frac{2}{i_1+2}\right\} - \sum_{i_1=1}^{n-1}\frac{1}{(i_1+1)^2} = 1 - \frac{2}{n+1} - \left(\frac{\pi^2}{6} - 1 - \sum_{i_1=n}^{\infty}\frac{1}{(i_1+1)^2}\right)\\ &= 1 - \frac{\pi^2}{6} + \frac{n^2}{(n+1)^2} + \sum_{i_1=n+1}^{\infty}\frac{1}{(i_1+1)^2} = 1 - \frac{\pi^2}{6} + \frac{n^2}{(n+1)^2} + \psi'(n+2),\end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Thus, in addition to the previous condition, those sequences under consideration should also admit for each $p = 2, 3, \ldots$ a closed-form expression for the n^{th} partial sum of

$$a_{i_1+p-1}\left\{\sum_{i_2=1}^{i_1} \cdots \sum_{i_p=1}^{i_{p-1}} a_{i_2+p-2}\cdots a_{i_p}\right\}.$$

Recalling that the partial sum of an arithmetic progression can be expressed as a linear combination of at most two binomial coefficients, we observe from the following result (see [3]) that $a_n = a_1 + (n-1)d$ (where a_1, $d \in \mathbb{C}$) is one such sequence that satisfies the required properties.

Lemma 3.1: Let $r \in \mathbb{N}^+$, then

$$\sum_{i=1}^{n}\binom{i+r}{r+1} = \binom{n+r+1}{r+2},$$

$$\sum_{i=1}^{n} i\binom{i+r}{r+1} = (r+2)\binom{n+r+2}{r+3} - (r+1)\binom{n+r+1}{r+2}.$$

Therefore, with the aid of Lemma 3.1, we can now state and prove the desired expansion theorem.

Theorem 3.1: Suppose $\{a_n\}$ is an arithmetic progression where $a_n = a_1 + (n-1)d$ for a given a_1, $d \in \mathbb{C}$. Then the equation for the coefficient of x^{n-p} for $n = 2, 3, \ldots$ and $p = 1, 2, \ldots, n$ in the resulting polynomial expansion of $(x)_{a_n}$ consists of a linear combination of binomial coefficients as follows,

$$\phi_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+p+1-m}{2p+1-m}, \tag{13}$$

in which the corresponding scalars $\theta_m^{(p)}$ are determined via the accompanying algorithm.

Algorithm 3.1: Set $\theta_1^{(1)} = d$ and $\theta_2^{(1)} = a_1 - d$, then calculate remaining scalars $\theta_m^{(p)}$ iteratively as follows:

for $i = 2, 3, \ldots, n$,
  $\theta_1^{(i)} = \theta_1^{(i-1)} d(2i-1)$;
for $j = 2, 3, \ldots, i$,
  $\theta_j^{(i)} = \theta_{j-1}^{(i-1)}(d(j-i-2)+a_1) + \theta_j^{(i-1)}(2i-j)d$;
  $\theta_{i+1}^{(i)} = \theta_i^{(i-1)}(a_1 - d)$.

Proof: In Proposition 2.1, let $a_n = a_1 + (n-1)d$ and set $\tilde{\phi}_p(n) = \phi_p(n+p-1)$, noting that in the resulting p-fold summation for $\tilde{\phi}_p(n)$ we no longer need require $p \le n$. Thus, it suffices to demonstrate via the following inductive argument on the parameter p, that there exists $\theta_m^{(p)} \in \mathbb{C}$ such that

$$\tilde{\phi}_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+2p-m}{2p+1-m}. \tag{14}$$

Beginning with $p = 1$, we have

$$\tilde{\phi}_1(n) = \sum_{i_1=1}^{n} (a_1 - d) + i_1 d = \sum_{m=1}^{2} \theta_m^{(1)} \binom{n+2-m}{3-m},$$

where $\theta_1^{(1)} = d$ and $\theta_2^{(1)} = a_1 - d$; consequently, (14) is valid for $p = 1$. Assume now that the result holds for $p = k$ where $k \ge 1$. To facilitate the inductive step, consider from (7) the expression for $\tilde{\phi}_{k+1}(n)$ as follows:

$$\tilde{\phi}_{k+1}(n) = \sum_{i_1=1}^{n} a_{i_1+k} \left\{ \sum_{i_2=1}^{i_1} \cdots \sum_{i_{k+1}=1}^{i_k} a_{i_2+k-1} \cdots a_{i_{k+1}} \right\}.$$

If necessary, by relabeling index variables, observe that the k-fold summation within the parentheses of the above equation, is equal to $\tilde{\phi}_k(i_1)$. Therefore, by assumption, we have

$$\tilde{\phi}_{k+1}(n) = \sum_{i_1=1}^{n} a_{i_1+k} \left\{ \sum_{m=1}^{k+1} \theta_m^{(k)} \binom{i_1+2k-m}{2k+1-m} \right\} = \sum_{m=1}^{k+1} \theta_m^{(k)} \left\{ \sum_{i_1=1}^{n} a_{i_1+k} \binom{i_1+2k-m}{2k+1-m} \right\}. \tag{15}$$

Now, for each m, an application of Lemma 3.1 yields

$$\begin{aligned}\sum_{i_1=1}^{n} a_{i_1+k} \binom{i_1+2k-m}{2k+1-m} &= a_k \sum_{i_1=1}^{n} \binom{i_1+2k-m}{2k+1-m} + d \sum_{i_1=1}^{n} i_1 \binom{i_1+2k-m}{2k+1-m} \\ &= \alpha(k,m) \binom{n+2k+1-m}{2k+2-m} + \beta(k,m) \binom{n+2k+2-m}{2k+3-m},\end{aligned}$$

where $\alpha(k,m) = d(m-k-2) + a_1$ and $\beta(k,m) = d(2k-m+2)$. As a result, (15) reduces to

$$\tilde{\phi}_{k+1}(n)=\sum_{m=1}^{k+1}\theta_m^{(k)}\beta(k,m)\binom{n+2k+2-m}{2k+3-m}+\sum_{m=1}^{k+1}\theta_m^{(k)}\alpha(k,m)\binom{n+2k+1-m}{2k+2-m}.$$

Finally, since

$$\sum_{m=1}^{k+1}\theta_m^{(k)}\alpha(k,m)\binom{n+2k+1-m}{2k+2-m}=\sum_{m=2}^{k+2}\theta_{m-1}^{(k)}\alpha(k,m-1)\binom{n+2k+2-m}{2k+3-m},$$

we deduce that

$$\tilde{\phi}_{k+1}(n)=\sum_{m=1}^{k+2}\theta_m^{(k+1)}\binom{n+2k+2-m}{2k+3-m},$$

where

$$\theta_1^{(k+1)}=\theta_1^{(k)}\beta(k,1), \tag{16}$$

$$\theta_m^{(k+1)}=\theta_m^{(k)}\beta(k,m)+\theta_{m-1}^{(k)}\alpha(k,m-1)\ \text{ for } m=2,3,\ldots,k+1, \tag{17}$$

$$\theta_{k+2}^{(k+1)}=\theta_{k+1}^{(k)}\alpha(k,k+1). \tag{18}$$

Hence, the result holds for $p=k+1$. and so, by induction, (14) is valid for all $p=1,2,\ldots$. Having established an explicit equation for $\phi_p(n)$, we note that it may be extended to encompass the case $p=0$ by defining $\theta_1^{(0)}\equiv 1$. It is now a simple matter to construct the accompanying algorithm. We begin by arranging those scalars involved in the first $n+1$ coefficients into a lower-triangular matrix as follows:

$$A_n=\begin{bmatrix}\theta_1^{(0)} & 0 & 0 & \cdots & 0\\ \theta_1^{(1)} & \theta_2^{(1)} & 0 & \cdots & 0\\ \theta_1^{(2)} & \theta_2^{(2)} & \theta_3^{(2)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \theta_1^{(n)} & \theta_2^{(n)} & \theta_3^{(n)} & \cdots & \theta_{n+1}^{(n)}\end{bmatrix}.$$

Suppose it is required that the n rows of A_n are to be determined. Clearly, from above, the entries of row two are given by $\theta_1^{(1)}=d$ and $\theta_2^{(1)}=a_1-d$. Now let us assume for argument's sake that the $(i-1)^{\text{th}}$ row has been calculated where $i\geq 2$. Then the following row of values can be obtained from the former by setting $k=i-1$ in equations (16), (17), and (18). Consequently, we deduce from (16) that

$$\theta_1^{(i)}=\theta_1^{(i-1)}d(2i-1). \tag{19}$$

While (17) implies, for $j=2,3,\ldots,i$,

$$\theta_j^{(i)}=\theta_j^{(i-1)}\beta(i-1,j)+\theta_{j-1}^{(i-1)}\alpha(i-1,j-1), \tag{20}$$

where $\alpha(i-1,j-1)=d(j-i-2)+a_1$ and $\beta(i-1,j)=d(2i-j)$. Similarly, from (18),

$$\theta_{i+1}^{(i)}=\theta_i^{(i-1)}(a_1-d). \tag{21}$$

Then, clearly, as the initial two rows of values are known, we may calculate all remaining $n-1$ rows by applying equations (19), (20), and (21) in succession for each $i=2,3,\ldots,n$, thus completing A_n. The algorithm now readily follows. □

The binomial theorem will now follow from Theorem 3.1 by demonstrating that, for a constant sequence (say $a_n = a$), the matrix A_n is rendered diagonal with $\theta_p^{(p+1)} = a^p$ for $p = 0, 1, \ldots, n$.

Corollary 3.1: Let $a, b \in \mathbb{C}$. Then, for all integers $n \geq 1$,

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}. \tag{22}$$

Proof: In what follows, assume $n \geq 2$, as (22) holds trivially for $n = 1$. Consider an arithmetic progression defined by $a_1 = a$ and $d = 0$. Then, by Theorem 3.1, the coefficient of x^{n-p} for $p = 0, 1, \ldots, n$ in the polynomial expansion of $(x+a)^n$ is equal to

$$\phi_p(n) = \sum_{m=1}^{p+1} \theta_m^{(p)} \binom{n+p+1-m}{2p+1-m}.$$

We assert that $\theta_j^{(i)} = 0$ for every $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, i$. From Algorithm 3.1, it is clear that $\theta_1^{(1)} = 0$. Now, if the result is assumed to hold for the $(i-1)^{\text{th}}$ row where $2 \leq i \leq n$, then $\theta_1^{(i)} = \theta_1^{(i-1)} d(2i-1) = 0$, while $\theta_j^{(i)} = \theta_{j-1}^{(i-1)} a = 0$ for $j = 2, 3, \ldots, i$. Hence, via the principle of finite mathematical induction the assertion is valid, and so

$$\phi_p(n) = \theta_{p+1}^{(p)} \binom{n}{p}$$

for $p = 0, 1, \ldots, n$. Now, as $\theta_2^{(1)} = a$ and $\theta_{i+1}^{(i)} = a\theta_i^{(i-1)}$ for $i = 2, 3, \ldots, n$, a similar inductive argument establishes $\theta_{p+1}^{(p)} = a^p$ for $p = 1, 2, \ldots, n$. Consequently, by recalling that $\theta_1^{(0)} \equiv 1$, we deduce that the coefficient of x^{n-p} for $p = 0, 1, \ldots, n$ in the above expansion is equal to

$$a^p \binom{n}{p}.$$

Setting $x = b$ yields the statement of the binomial theorem for the given n; however, the result now follows as this was arbitrarily chosen. □

To contrast the previous result, we shall consider now an alternate expansion theorem for $(x)_{a_n}$ where a_n is a geometric progression (i.e., $a_n = az^n$ for $a, z \in \mathbb{C}$); however, unlike Theorem 3.1, no algorithm will be required to complete the formulation of $\phi_p(n)$. It should be noted that setting $z = 1$ in this result will not produce the binomial theorem, as the expressions for $\phi_p(n)$ in this case reduce to an indeterminate form.

Theorem 3.2: For a given integer $n = 2, 3, \ldots$, set $I_n = \{z \in \mathbb{C} \mid z = \sqrt[r]{1},\ r = 1, 2, \ldots, n\}$. If $a_n = az^n$ with $a, z \in \mathbb{C}$ and $z \notin I_n$, then the coefficient of x^{n-p} for $p = 1, 2, \ldots, n$ in the polynomial expansion of $(x)_{a_n}$ is given by

$$\phi_p(n) = a^p z^{\frac{1}{2}p(p+1)} \prod_{j=1}^{p} \frac{1 - z^{n-j+1}}{1 - z^j}. \tag{23}$$

Proof: The result clearly holds for $p = 1$ as $\phi_1(n)$ is equal to the n^{th} partial sum of $\{a_n\}$. To demonstrate (23) for $p = 2, 3, \ldots, n$, consider for a fixed $z \notin I_n$ the polynomial function in x,

$$p_n(x) = (1+zx)(1+z^2x)\cdots(1+z^nx) = \sum_{m=0}^{n} c_m x^m.$$

Clearly, $p_n(x)$ satisfies the following functional identity:

$$(1+z^{n+1}x)p_n(x) = (1+zx)p_n(zx). \tag{24}$$

Substituting the above partial sum for $p_n(x)$ in (24) and equating coefficients of x^m yields

$$c_m + c_{m-1}z^{n+1} = (c_m + c_{m-1})z^m,$$

where $m = 1, 2, \ldots, n$. Thus, after some rearrangement of terms, we obtain the recurrence relation

$$c_m = \frac{z^m(1-z^{n-m+1})}{1-z^m}c_{m-1},$$

from which one easily deduces, as $c_0 = 1$, the formula

$$c_p = z^{\frac{1}{2}p(p+1)}\prod_{j=1}^{p}\frac{1-z^{n-j+1}}{1-z^j}, \tag{25}$$

where $p = 2, 3, \ldots, n$, noting here that the expression in (25) is well-defined due to the restriction $z \notin I_n$. Now let $x = y^{-1}$ for $y \neq 0$ and observe that $y^n p_n(y^{-1}) = (y)_{a_n}$, where $a_n = z^n$; hence, $\phi_p(n) = c_p$. Therefore, by Proposition 2.1, we find that

$$c_p = z^{\frac{1}{2}p(p-1)}\sum_{i_1=1}^{n-p+1}\sum_{i_2=1}^{i_1}\cdots\sum_{i_p=1}^{i_{p-1}} z^{i_1+i_2+\cdots+i_p} \tag{26}$$

for $p = 2, 3, \ldots, n$. As the coefficient of x^{n-p} for $p = 2, 3, \ldots, n$ in $(x)_{a_n}$, where $a_n = az^n$ is equal to $a^p c_p$, we deduce from (25) the desired expression. □

It is possible to retrieve a binomial coefficient from the expression in (23) by taking the limit as $z \to w$, where w is a root of unity. The result that follows may be obtained by an application of L'Hôpital's rule for indeterminant forms; however, the argument used below is probably more direct. We will require the following technical lemma.

Lemma 3.2: If w is a primitive m^{th} root of unity, where m is a positive even integer, then

$$(x+w)(x+w^2)\cdots(x+w^m) = x^m - 1.$$

Proof: Let $a_n = w^n$ and consider the polynomial $(x)_{a_m}$. By making the substitution $x = -y$, observe that

$$(-y)_{a_m} = (-1)^m\prod_{j=1}^{m}(y-w^j) = \prod_{j=1}^{m}(y-w^j).$$

Since w is a primitive root of unity, the set $\{w, w^2, \ldots, w^m\}$ contains all the m^{th} roots of unity without repetition. Hence, the product on the right of the above is equal to $y^m - 1 = x^m - 1$. □

Corollary 3.2: Suppose n, m, and p are positive integers with m even and $1 \le p \le mn$. If w is a primitive m^{th} root of unity, then the following limit holds:

$$\lim_{z\to w}\prod_{j=1}^{p}\frac{1-z^{mn-j+1}}{1-z^{j}}=\begin{cases}\frac{(-1)^s}{w^{\frac{ms}{2}}}\binom{n}{s} & \text{for } p=ms,\\ 0 & \text{for } p\neq ms.\end{cases}$$

Proof: Set $a_n = z^n$ and consider for a fixed x the polynomial in z and x given by

$$f(z)=(x)_{a_{mn}}=(x+z)(x+z^2)\cdots(x+z^{mn}).$$

As $f(z)$ is a continuous function of the complex variable z, we have

$$\lim_{z\to w} f(z)=f(w)=(x^m-1)^n=\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}x^{mr}, \tag{27}$$

noting here that the right-hand side follows from Lemma 3.2 and the periodicity of the sequence $\{w^n\}$. Now when $z\notin I_{mn}$ one can expand $f(z)$ in a polynomial in x as follows,

$$f(z)=\sum_{p=0}^{mn}\phi_p(mn)x^{mn-p}, \tag{28}$$

where the complex coefficients $\phi_p(mn)$ are of the form as stated in Theorem 3.2. As the set I_{mn} contains only finitely many complex numbers, there must exist, for $\delta>0$ sufficiently small, an open neighborhood about w of the form $B_\delta(w):=\{z\in\mathbb{C}:|z-w|<\delta\}$ such that $B_\delta(w)\cap I_{mn}=\{w\}$. Hence, the expression for $f(z)$ in (28) is valid in the deleted neighborhood $B_\delta(w)\backslash\{w\}$ and so, by (27)

$$\lim_{z\to w}\sum_{p=0}^{mn}\phi_p(mn)x^{mn-p}=\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}x^{mr}.$$

Clearly, as the right-hand side of the above consists of a polynomial in x^m, we have

$$\lim_{z\to w}\frac{\phi_p(mn)}{z^{\frac{1}{2}p(p+1)}}=0 \quad\text{when}\quad p\neq ms,$$

while, if $p=ms$, then by setting $r=n-s$ one deduces again from (27) that

$$\lim_{z\to w}\frac{\phi_p(mn)}{z^{\frac{1}{2}p(p+1)}}=\frac{(-1)^s}{w^{\frac{1}{2}ms(ms+1)}}\binom{n}{n-s}=\frac{(-1)^s}{w^{\frac{ms}{2}}}\binom{n}{s}. \quad\square$$

Remark 3.2: In the case in which $m=4$, we have for $w=\pm i$ the limit

$$\lim_{z\to w}\prod_{j=1}^{4s}\frac{1-z^{4n-j+1}}{1-z^{j}}=\binom{n}{s}$$

where $1\le s\le n$.

4. APPLICATION

We now turn our attention to the Pochhammer class of polynomial functions which result from (3) by setting $a_n=n-1$. This family of polynomials was first studied by Stirling in 1730 and later by Appell; however, the name Pochhammer is used in recognition for the invention of the symbol $(x)_n$. These polynomials feature in many areas of analysis, including the study of special functions, where they occur in the coefficients of hypergeometric series (see [5], p. 149). When expanded into a polynomial, $(x)_n$ can be written as

$$(x)_n = \sum_{r=0}^{n} |S_r^{(n)}| x^r,$$

where the integers $S_r^{(n)}$ are the Stirling numbers of first order. This group of numbers are normally calculated by first defining $S_0^{(n)} = 0$, $S_n^{(n)} = 1$, and then applying the recursion formula $S_n^{(r)} = S_{n-1}^{(r-1)} - rS_{n-1}^{(r)}$ for each $n = 1, 2, \ldots$ and $r = 1, 2, \ldots, n$ in succession. However, the Stirling numbers $S_{n-p}^{(n)}$ for $p = 1, 2, \ldots$ also appear as the coefficients of x^{n-p}, in the falling factorial polynomial of degree n which results from (3) by setting $a_n = 1 - n$ (see [5], p. 20). Thus, by applying Theorem 3.1 with the parameters $a_1 = 0$ and $d = -1$, we can now derive algebraic expressions for $S_{n-p}^{(n)}$. To illustrate, suppose three iterations of Algorithm 3.1 are performed, thereby producing the matrix

$$A_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ -15 & 25 & -11 & 1 \end{bmatrix}.$$

Then, by reading directly from this matrix we deduce, using (13), the following formulas:

$$S_{n-1}^{(n)} = \binom{n}{1} - \binom{n+1}{2},$$

$$S_{n-2}^{(n)} = \binom{n}{2} - 4\binom{n+1}{3} + 3\binom{n+2}{4},$$

$$S_{n-3}^{(n)} = \binom{n}{3} - 11\binom{n+1}{4} + 25\binom{n+2}{5} - 15\binom{n+3}{6}.$$

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