EVEN DUCCI-SEQUENCES

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Ducci-sequences are successive iterations of the function

$$D(X) = D(x_1, x_2, \ldots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1|).$$

Note that $D: Z^n \to Z^n$, where Z^n is the set of n-tuples with integer entries. Since the entries of $D(X)$ are less than or equal to those of X, eventually every Ducci-sequence $\{X, D(X), D^2(X), \ldots, D^j(X), \ldots\}$ gives rise to a cycle. That is, there exist integers i and j for which $0 \le i < j$ and $D^j(X) = D^i(X)$. When i and j are as small as possible, we say that the resulting cycle, $\{D^i(X), \ldots, D^{j-1}(X), \ldots\}$, is generated by X and has period $j - i$. If Y is contained in a cycle of period k, then $D^j(Y) = Y$ if and only if $k \mid j$.

Introduced in 1937, Ducci-sequences and their resulting cycles have been studied extensively (see [1]-[7]). It is well known that for a given cycle all the entries in all the tuples are equal to either 0 or a constant C (see [2] and [4]). Since for every λ, $D(\lambda X) = \lambda D(X)$, we can assume without loss of generality that $C = 1$. Thus, when studying cycles of Ducci-sequences, we can restrict our attention to Z_2^n, the set of n-tuples with entries from $\{0, 1\}$. In addition, we can view the operation associated with D as addition modulo 2 since, for $x, y \in \{0, 1\}$, $|x - y| \equiv (x + y) \pmod 2$.

Most of the work on Ducci-sequences has focused on the case when n is odd or a power of 2. Here we consider the case when $n = 2^s \cdot q$, where $s \ge 1$ and q is odd with $q > 1$. We will show that associated with an n-tuple X are 2^s different q-tuples that completely determine the behavior of X. In particular, we will show that an n-tuple X is contained in a cycle if and only if each of the 2^s associated q-tuples is in a cycle. Further, the period of the cycle generated by X is determined by the periods of the cycles generated by the 2^s associated q-tuples.

To motivate the notation that will be introduced shortly, consider the following representations of a 12-tuple X:

$$\begin{aligned} X &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}) \\ &= (x_1, 0, 0, 0, x_5, 0, 0, 0, x_9, 0, 0, 0) + (0, x_2, 0, 0, 0, x_6, 0, 0, 0, x_{10}, 0, 0) \\ &\quad + (0, 0, x_3, 0, 0, 0, x_7, 0, 0, 0, x_{11}, 0) + (0, 0, 0, x_4, 0, 0, 0, x_8, 0, 0, 0, x_{12}). \end{aligned}$$

We see that associated with X are the following four 3-tuples:

$$(x_1, x_5, x_9),\ (x_2, x_6, x_{10}),\ (x_3, x_7, x_{11}),\ \text{and } (x_4, x_8, x_{12}).$$

When we form these smaller tuples, we will say that we compress the original tuple. Conversely, we can begin with these four 3-tuples and expand them to a 12-tuple by inserting zeros and adding.

Since we are interested in even tuples, we will often need to work with powers of 2. To simplify notation, we will write 2^s as $2\wedge s$ whenever this expression appears as a superscript or subscript.

Let X be an n-tuple where $n = 2^s \cdot q$ with $s \geq 1$. For $i \in \{1, 2, \ldots, 2^s\}$, the compression functions $C_{i,2^\wedge s}: Z_2^n \to Z_2^q$ are defined by $C_{i,2^\wedge s}(X) = (c_j)$, where

$$c_j = x_{i+(j-1)} \cdot 2^\wedge s.$$

For $i \in \{1, 2, \ldots, 2^s\}$, the expansion functions $E_{i,2^\wedge s}: Z_2^q \to Z_2^n$ are defined by $E_{i,2^\wedge s}(Y) = (e_j)$, where

$$\begin{cases} e_j = y_{\lambda+1} & \text{when } j = i + \lambda \cdot 2^s \text{ for } \lambda = 0, 1, \ldots, q-1, \\ e_j = 0 & \text{when } j \not\equiv i \pmod{2^s}. \end{cases}$$

The observations below follow immediately from the definitions of the compression and expansion functions:

$$X = \sum_{i=1}^{2^\wedge s} E_{i,2^\wedge s}(C_{i,2^\wedge s}(X)) \text{ for } X \in Z_2^n, \text{ where } 2^s \mid n; \tag{1}$$

$$C_{i,2}(E_{i,2}(Y)) = Y \text{ for } Y \in Z_2^q, \text{ where } n = 2 \cdot q; \tag{2}$$

$$C_{i,2}(E_{j,2}(Y)) = (0, 0, \ldots, 0) \text{ for } Y \in Z_2^q, \text{ where } n = 2 \cdot q \text{ and } i \neq j; \tag{3}$$

$$C_{j,2}(C_{i,2^\wedge s}(X)) = C_{i+(j-1)\cdot 2^\wedge s,\, 2^\wedge(s+1)}(X) \text{ for } X \in Z_2^n, \text{ where } 2^{s+1} \mid n; \tag{4}$$

$$E_{i,2^\wedge s}(E_{j,2}(Y)) = E_{i+(j-1)\cdot 2^\wedge s,\, 2^\wedge(s+1)}(Y) \text{ for } Y \in Z_2^q, \text{ where } n = 2^{s+1} \cdot q. \tag{5}$$

We use these observations to express $D^{2^\wedge s \cdot m}(X)$ in terms of $D^m(C_{i,2^\wedge s}(X))$.

Theorem 1: Let X be an n-tuple, where $2 \mid n$. Then

$$D^2(X) = \sum_{i=1}^{2} E_{i,2}(D(C_{i,2}(X))).$$

Proof: Let $X = (x_1, x_2, \ldots, x_n)$. Then

$$D(X) = (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4,\ x_4 + x_5, \ldots, x_{n-1} + x_n,\ x_n + x_1),$$

$$D^2(X) = (x_1 + x_3,\ x_2 + x_4,\ x_3 + x_5,\ x_4 + x_6, \ldots, x_{n-1} + x_1,\ x_n + x_2).$$

On the other hand,

$$C_{1,2}(X) = (x_1, x_3, x_5, \ldots, x_{n-1}),$$

$$D(C_{1,2}(X)) = (x_1 + x_3,\ x_3 + x_5, \ldots, x_{n-1} + x_1),$$

$$E_{1,2}(D(C_{1,2}(X))) = (x_1 + x_3,\ 0,\ x_3 + x_5,\ 0, \ldots, x_{n-1} + x_1,\ 0).$$

Similarly,

$$E_{2,2}(D(C_{2,2}(X))) = (0,\ x_2 + x_4,\ 0,\ x_4 + x_6, \ldots, 0,\ x_n + x_2).$$

Thus

$$D^2(X) = E_{1,2}(D(C_{1,2}(X))) + E_{2,2}(D(C_{2,2}(X))). \quad \square$$

Theorem 2: Let X be an n-tuple, where $2|n$. Then

$$D^{2m}(X)=\sum_{i=1}^{2}E_{i,2}(D^m(C_{i,2}(X))).$$

Proof: By Theorem 1, the result holds for $m=1$. Assume it holds for m and consider $m+1$. Now $D^{2(m+1)}(X)=D^2(D^{2m}(X))$. Thus

$$D^{2(m+1)}(X)=D^2\left(\sum_{i=1}^{2}E_{i,2}(D^m(C_{i,2}(X)))\right)=\sum_{j=1}^{2}E_{j,2}\left(D\left(C_{j,2}\left(\sum_{i=1}^{2}E_{i,2}(D^m(C_{i,2}(X)))\right)\right)\right). \quad (6)$$

Using observations (2) and (3), (6) simplifies to

$$\begin{aligned}D^{2(m+1)}(X) &= E_{1,2}(D(D^m(C_{1,2}(X))))+E_{2,2}(D(D^m(C_{2,2}(X))))\\ &=\sum_{i=1}^{2}E_{i,2}(D^{m+1}(C_{i,2}(X))). \quad \square\end{aligned}$$

Theorem 3: Let X be an n-tuple, where $2^s|n$ with $s\ge 1$. Then

$$D^{2\wedge s\cdot m}(X)=\sum_{i=1}^{2\wedge s}E_{i,2\wedge s}(D^m(C_{i,2\wedge s}(X))).$$

Proof: By Theorem 2, the result holds for $s=1$. Assume it holds for s and consider $s+1$. Using the induction hypothesis, we have

$$\begin{aligned}D^{2\wedge(s+1)\cdot m}(X)=D^{(2\wedge s)\cdot(2m)}(X)&=\sum_{i=1}^{2\wedge s}E_{i,2\wedge s}(D^{2m}(C_{i,2\wedge s}(X)))\\ &=\sum_{i=1}^{2\wedge s}E_{i,2\wedge s}\left(\sum_{j=1}^{2}E_{j,2}(D^m(C_{j,2}(C_{i,2\wedge s}(X))))\right).\end{aligned} \quad (7)$$

The last equality in (7) follows from Theorem 2. Using observations (4) and (5), (7) simplifies to

$$\begin{aligned}D^{2\wedge(s+1)\cdot m}(X) &=\sum_{i=1}^{2\wedge s}\sum_{j=1}^{2}E_{i+(j-1)\cdot 2\wedge s,\,2\wedge(s+1)}(D^m(C_{i+(j-1)\cdot 2\wedge s,\,2\wedge(s+1)}(X)))\\ &=\sum_{i=1}^{2\wedge(s+1)}E_{i,2\wedge(s+1)}(D^m(C_{i,2\wedge(s+1)}(X))). \quad \square\end{aligned}$$

Corollary 1: Let X be an n-tuple, where $2^s|n$;with $s\ge 1$. X is contained in a cycle if and only if $C_{i,2\wedge s}(X)$ is contained in a cycle for $i\in\{1,\dots,2^s\}$.

Proof: Suppose X is contained in a cycle of period k; that is, $D^k(X)=X$. Then

$$D^{2\wedge s\cdot k}(X)=X.$$

Using (1) and Theorem 3, we see that

$$C_{i,2\wedge s}(X)=C_{i,2\wedge s}(D^{2\wedge s\cdot k}(X))=D^k(C_{i,2\wedge s}(X))$$

for $i\in\{1,\dots,2^s\}$. Hence for each i, $C_{i,2\wedge s}(X)$ is in a cycle.

Conversely, suppose that, for each i, $C_{i,2^\wedge s}(X)$ is in a cycle of period k_i. Let $m = \text{lcm}(k_1, k_2, \ldots, k_{2^\wedge s})$. Since $D^m(C_{i,2^\wedge s}(X)) = C_{i,2^\wedge s}(X)$, by Theorem 3 and (1), $D^{2^\wedge s \cdot m}(X) = X$. Hence, X is in a cycle. □

For odd n, an n-tuple X is contained in a cycle if and only if the sum of the entries of X is congruent to 0 modulo 2 (see [4]). Thus by Corollary 1, for $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$, an n-tuple X is contained in a cycle if and only if for each $i \in \{1, \ldots, 2^s\}$ the sum of the entries of $C_{i,2^\wedge s}(X)$ is congruent to 0 modulo 2. Although the terminology is different, this result appears in [4]. In a moment we will begin to consider how the period of the cycle containing X is related to the periods of the cycles containing $C_{i,2^\wedge s}(X)$, $i = 1, \ldots, 2^s$. First, we prove a rather technical corollary that we will need later.

Corollary 2: Let X be an n-tuple, where $2^s | n$ with $s \geq 1$. Then

$$C_{i,2^\wedge s}(D^{2^\wedge(s-1)}(X)) = C_{i,2^\wedge s}(X) + C_{i+2^\wedge(s-1),2^\wedge s}(X)$$

for $i = 1, 2, \ldots, 2^{s-1}$.

Proof: Let $n = 2^s \cdot q = 2^{s-1} \cdot 2q$. By Theorem 3,

$$D^{2^\wedge(s-1)}(X) = \sum_{i=1}^{2^\wedge(s-1)} E_{i,2^\wedge(s-1)}(D(C_{i,2^\wedge(s-1)}(X))).$$

For $Z \in Z_2^{2q}$ and $i = 1, 2, \ldots, 2^{s-1}$,

$$C_{i,2^\wedge s}(E_{i,2^\wedge(s-1)}(Z)) = C_{1,2}(Z),$$
$$C_{j,2^\wedge s}(E_{i,2^\wedge(s-1)}(Z)) = (0, 0, \ldots, 0) \text{ when } j \neq i.$$

Hence $C_{i,2^\wedge s}(D^{2^\wedge(s-1)}(X)) = C_{1,2}(D(C_{i,2^\wedge(s-1)}(X)))$. Now

$$C_{i,2^\wedge(s-1)}(X) = (x_i, x_{i+2^\wedge(s-1)}, x_{i+2\cdot 2^\wedge(s-1)}, x_{i+3\cdot 2^\wedge(s-1)}, \ldots, x_{i+(2q-1)\cdot 2^\wedge(s-1)}),$$

$$D(C_{i,2^\wedge(s-1)}(X)) = (x_i + x_{i+2^\wedge(s-1)} + x_{i+2\cdot 2^\wedge(s-1)}, x_{i+2\cdot 2^\wedge(s-1)}, \ldots, x_{i+(2q-1)\cdot 2^\wedge(s-1)} + x_i),$$

$$\begin{aligned} C_{i,2^\wedge s}(D^{2^\wedge(s-1)}(X)) &= C_{1,2}(D(C_{i,2^\wedge(s-1)}(X))) \\ &= (x_i + x_{i+2^\wedge(s-1)}, x_{i+2\cdot 2^\wedge(s-1)} + x_{i+3\cdot 2^\wedge(s-1)}, \ldots, x_{i+(2q-2)\cdot 2^\wedge(s-1)} + x_{i+(2q-1)\cdot 2^\wedge(s-1)}) \\ &= (x_i, x_{i+2\cdot 2^\wedge(s-1)}, \ldots, x_{i+(2q-2)\cdot 2^\wedge(s-1)}) + (x_{i+2^\wedge(s-1)}, x_{i+3\cdot 2^\wedge(s-1)}, \ldots, x_{i+(2q-1)\cdot 2^\wedge(s-1)}) \\ &= C_{i,2^\wedge s}(X) + C_{i+2^\wedge(s-1),2^\wedge s}(X). \quad \square \end{aligned}$$

We now begin considering how the period of the cycle containing X is related to the periods of the cycles containing $C_{i,2^\wedge s}(X)$, $i = 1, \ldots, 2^s$.

Theorem 4: Let $n = 2^s \cdot q$, where $s \geq 1$. Suppose X is an n-tuple which is contained in a cycle of period k. Let k_i be the period of the cycle containing the q-tuple $C_{i,2^\wedge s}(X)$, $i = 1, \ldots, 2^s$. Then $k = 2^t \cdot \text{lcm}(k_1, k_2, \ldots, k_{2^\wedge s})$ for some $0 \leq t \leq s$.

Proof: Let $m = \text{lcm}(\text{k}_1, k_2, \ldots, k_{2^\wedge s})$. As noted in the proof of Corollary 1, $D^{2^\wedge s \cdot m}(X) = X$. Consequently, $k \,|\, 2^s \cdot m$.

We now show that $m|k$. Since $D^k(X)=X$, it follows that $D^{2^\wedge s\cdot k}(X)=X$. As we showed in the proof of Corollary 1, $D^k(C_{i,2^\wedge s}(X))=C_{i,2^\wedge s}(X)$. Since k_i is the period of the cycle containing $C_{i,2^\wedge s}(X)$, $k_i|k$ for $i=1,\ldots,2^s$. Consequently, $m|k$. Since $m|k$ and $k|2^s\cdot m$, we conclude that $k=2^t\cdot m$ for some $0\le t\le s$. □

Theorem 5: Let $n=2^s\cdot q$, where $s\ge 1$. Suppose X is an n-tuple which is contained in a cycle of period $k=2^t\cdot m$, where m is odd and $0\le t<s$. Then

$$C_{i+2^\wedge t,\,2^\wedge(t+1)}(X)=C_{i,\,2^\wedge(t+1)}(X)+D^{\frac{m+1}{2}}(C_{i,\,2^\wedge(t+1)}(X))$$

for $i=1,\ldots,2^t$.

Proof: Since $0\le t<s$, $1\le t+1\le s$, and $2^{t+1}|n$. Thus by Theorem 3,

$$\begin{aligned}D^{2^\wedge t\cdot(m-1)}(X)&=D^{2^\wedge(t+1)\cdot\frac{m-1}{2}}(X)\\&=\sum_{i=1}^{2^\wedge(t+1)}E_{i,\,2^\wedge(t+1)}\left(D^{\frac{m-1}{2}}(C_{i,\,2^\wedge(t+1)}(X))\right).\end{aligned}\tag{8}$$

By hypothesis, $D^{2^\wedge t\cdot m}(X)=X$. Since $X=D^{2^\wedge t\cdot m}(X)=D^{2^\wedge t}(D^{2^\wedge t\cdot(m-1)}(X))$,

$$C_{i,\,2^\wedge(t+1)}(X)=C_{i,\,2^\wedge(t+1)}(D^{2^\wedge t}(D^{2^\wedge t\cdot(m-1)}(X))).$$

By Corollary 2,

$$C_{i,\,2^\wedge(t+1)}(D^{2^\wedge t}(D^{2^\wedge t\cdot(m-1)}(X)))=C_{i,\,2^\wedge(t+1)}(D^{2^\wedge t\cdot(m-1)}(X))+C_{i+2^\wedge t,\,2^\wedge(t+1)}(D^{2^\wedge t\cdot(m-1)}(X))$$

for $i=1,\ldots,2^t$. Thus

$$C_{i,\,2^\wedge(t+1)}(X)=C_{i,\,2^\wedge(t+1)}(D^{2^\wedge t\cdot(m-1)}(X))+C_{i+2^\wedge t,\,2^\wedge(t+1)}(D^{2^\wedge t\cdot(m-1)}(X)).\tag{9}$$

Using (8) to find the two terms on the right-hand side of (9), we can rewrite (9) as

$$C_{i,\,2^\wedge(t+1)}(X)=D^{\frac{m-1}{2}}(C_{i,\,2^\wedge(t+1)}(X))+D^{\frac{m-1}{2}}(C_{i+2^\wedge t,\,2^\wedge(t+1)}(X)).\tag{10}$$

Applying $D^{\frac{m-1}{2}}$ to (10) gives

$$D^{\frac{m-1}{2}}(C_{i,\,2^\wedge(t+1)}(X))=D^m(C_{i,\,2^\wedge(t+1)}(X))+D^m(C_{i+2^\wedge t,\,2^\wedge(t+1)}(X)).\tag{11}$$

By hypothesis, $D^{2^\wedge t\cdot m}(X)=X$. Hence $D^{2^\wedge(t+1)\cdot m}(X)=X$. Thus, using Theorem 3 and (1),

$$C_{j,\,2^\wedge(t+1)}(X)=C_{j,\,2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m}(X))=D^m(C_{j,\,2^\wedge(t+1)}(X))$$

for $j=1,\ldots,2^{t+1}$. Using this to simplify (11) and rearranging terms gives the desired result. □

We now prove the converse of Theorem 5. To do so, we will need the following well-known result: when n is odd, the period of a cycle of n-tuples divides $n\cdot(2^{\phi(n)}-1)$, where $\phi(n)$ is Euler's phi function [3]. Actually, a great deal more is known about the period, but this is all we require. Specifically, when n is odd, the period of each cycle of n-tuples is odd.

Theorem 6: Let $n=2^s\cdot q$, where $s\ge 1$ and q is odd with $q>1$. Suppose X is an n-tuple that is contained in a cycle. Let $m=\text{lcm}(k_1,k_2,\ldots,k_{2^\wedge s})$, where k_i is the period of the cycle containing $C_{i,2^\wedge s}(X)$ for $i=1,\ldots,2^s$. If there exists t, $0\le t<s$, such that

$$C_{i+2^\wedge t,\, 2^\wedge(t+1)}(X) = C_{i,\, 2^\wedge(t+1)}(X) + D^{\frac{m-1}{2}}(C_{i,\, 2^\wedge(t+1)}(X)) \tag{12}$$

for $i = 1, \ldots, 2^t$, then $D^{2^\wedge t \cdot m}(X) = X$.

Proof: Since q is odd, each k_i is odd and hence m is odd. Further, since $D^{k_i}(C_{i,\, 2^\wedge s}(X)) = C_{i,\, 2^\wedge s}(x)$, $D^m(C_{i,\, 2^\wedge s}(X)) = C_{i,\, 2^\wedge s}(X)$ for $i = 1, \ldots, 2^s$. Thus, if $t+1 = s$,

$$D^m(C_{i,\, 2^\wedge(t+1)}(X)) = C_{i,\, 2^\wedge(t+1)}(X).$$

On the other hand, if $r = t+1 < s$, then

$$\begin{aligned} C_{i,\, 2^\wedge r}(X) = C_{i,\, 2^\wedge s}(X) + C_{i+2^\wedge r,\, 2^\wedge s}(X) + C_{i+2\cdot 2^\wedge r,\, 2^\wedge s}(X) \\ + \cdots + C_{i+[2^\wedge(s-r)-1]\cdot 2^\wedge r,\, 2^\wedge s}(X). \end{aligned}$$

This implies $D^m(C_{i,\, 2^\wedge r}(X)) = C_{i,\, 2^\wedge r}(X)$; i.e., $D^m(C_{i,\, 2^\wedge(t+1)}(X)) = C_{i,\, 2^\wedge(t+1)}(X)$. Hence

$$C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m}(X)) = D^m(C_{i,\, 2^\wedge(t+1)}(X)) = C_{i,\, 2^\wedge(t+1)}(X),$$

so $D^{2^\wedge(t+1)\cdot m}(X) = X$. We now use this to show that, in fact, $D^{2^\wedge t\cdot m}(X) = X$.

As in the proof of Theorem 5, we consider $D^{2^\wedge t\cdot m}(X)$. Using (8), we have

$$C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge t\cdot(m-1)}(X)) = D^{\frac{m-1}{2}}(C_{i,\, 2^\wedge(t+1)}(X)). \tag{13}$$

Likewise, using (8) and (12), we have

$$\begin{aligned} C_{i+2^\wedge t,\, 2^\wedge(t+1)}(D^{2^\wedge t\cdot(m-1)}(X)) &= D^{\frac{m-1}{2}}(C_{i+2^\wedge t,\, 2^\wedge(t+1)}(X)) \\ &= D^{\frac{m-1}{2}}(C_{i,\, 2^\wedge(t+1)}(X)) + D^{\frac{m-1}{2}}(D^{\frac{m+1}{2}}(C_{i,\, 2^\wedge(t+1)}(X))) \\ &= D^{\frac{m-1}{2}}(C_{i,\, 2^\wedge(t+1)}(X)) + C_{i,\, 2^\wedge(t+1)}(X). \end{aligned} \tag{14}$$

Note that (13) and (14) hold for $i = 1, \ldots, 2^t$. Now, by Theorem 3, we have

$$C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot(m-1)}(X)) = D^{m-1}(C_{i,\, 2^\wedge(t+1)}(X)). \tag{15}$$

Likewise, using Theorem 3 and (12), we have

$$\begin{aligned} C_{i+2^\wedge t,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot(m-1)}(X)) &= D^{m-1}(C_{i+2^\wedge t,\, 2^\wedge(t+1)}(X)) \\ &= D^{m-1}(C_{i,\, 2^\wedge(t+1)}(X)) + D^{\frac{m-1}{2}}(C_{i,\, 2^\wedge(t+1)}(X)). \end{aligned} \tag{16}$$

Note that (15) and (16) hold for $i = 1, \ldots, 2^t$. By Corollary 2,

$$C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge t}(Z)) = C_{i,\, 2^\wedge(t+1)}(Z) + C_{i+2^\wedge t,\, 2^\wedge(t+1)}(Z). \tag{17}$$

We let $Z = D^{2^\wedge(t+1)\cdot(m-1)}(X)$ in (17), note that $2^t + 2^{t+1}\cdot(m-1) = 2^{t+1}\cdot m - 2^t$, and use (15) and (16) to get

$$C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m - 2^\wedge t}(X)) = D^{\frac{m-1}{2}}(C_{i,\, 2^\wedge(t+1)}(X)). \tag{18}$$

Now we let $Z = D^{2^\wedge(t+1)\cdot m - 2^\wedge t}(X)$ in (17). This gives us

$$\begin{aligned} C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m}(X)) = C_{i,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m - 2^\wedge t}(X)) \\ + C_{i+2^\wedge t,\, 2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m - 2^\wedge t}(X)). \end{aligned} \tag{19}$$

We rewrite (19) using (18) and the fact that $C_{i,2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m}(X))=C_{i,2^\wedge(t+1)}(X)$:

$$C_{i,2^\wedge(t+1)}(X)=D^{\frac{m-1}{2}}(C_{i,2^\wedge(t+1)}(X))+C_{i,2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m-2^\wedge t}(X))$$

or

$$C_{i+2^\wedge t,2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m-2^\wedge t}(X))=D^{\frac{m-1}{2}}(C_{i,2^\wedge(t+1)}(X))+C_{i,2^\wedge(t+1)}(X). \tag{20}$$

Comparing (13) to (18), we see that

$$C_{i,2^\wedge(t+1)}(D^{2^\wedge t\cdot m-2^\wedge t}(X))=C_{i,2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m-2^\wedge t}(X))$$

for $i=1,\ldots,2^t$, and comparing (14) to (20), we see that

$$C_{i+2^\wedge t,2^\wedge(t+1)}(D^{2^\wedge t\cdot m-2^\wedge t}(X))=C_{i+2^\wedge t,2^\wedge(t+1)}(D^{2^\wedge(t+1)\cdot m-2^\wedge t}(X))$$

for $i=1,\ldots,2^t$. Hence $D^{2^\wedge t\cdot m-2^\wedge t}(X)=D^{2^\wedge(t+1)\cdot m-2^\wedge t}(X)$. This, in turn, implies that $D^{2^\wedge t\cdot m}(X)=D^{2^\wedge(t+1)\cdot m}(X)=X$. □

Thus we have completely characterized the period of a cycle of n-tuples. We summarize the results of the last three theorems in the following corollary.

Corollary 3: Let $n=2^s\cdot q$, where $s\geq 1$ and q is odd with $q>1$. Suppose X is an n-tuple which is contained in a cycle of period k. Let $m=\operatorname{lcm}(k_1,k_2,\ldots,k_{2^\wedge s})$, where k_i is the period of the cycle containing $C_{i,2^\wedge s}(X)$. Then $k=2^t\cdot\operatorname{lcm}(k_1,k_2,\ldots,k_{2^\wedge s})$ for some $0\leq t<s$ if and only if

$$C_{i+2^\wedge t,2^\wedge(t+1)}(X)=C_{i,2^\wedge(t+1)}(X)+D^{\frac{m+1}{2}}(C_{i,2^\wedge(t+1)}(X))$$

for $i=1,\ldots,2^t$, where t is as small as possible. If no such t exists, then $k=2^s\cdot m$. □

We now show that there is a cycle for each possible period. Although there are many ways to do this, we will continue to use the compression functions.

Theorem 7: Let $n=2^s\cdot q$, where $s\geq 1$ and q is odd with $q>1$. Suppose there is a cycle of q-tuples of period m. Then, for $0\leq t\leq s$, there exists a cycle of n-tuples of period $2^t\cdot m$.

Proof: For $0\leq r\leq s-1$, suppose there is a $(2^{s-1}\cdot q)$-tuple A that is contained in a cycle of period $2^r\cdot m$. By hypothesis, this holds for $s=1$. Consider the $(2^s\cdot q)$-tuple $X=E_{1,2}(A)$. Now $C_{1,2}(X)=A$ and $C_{2,2}(X)=(0,0,\ldots,0)$. By Corollary 1, X is in a cycle. By Theorem 4, the period of the cycle containing X is either $2^r\cdot m$ or $2\cdot(2^r\cdot m)$. Assume the period is $2^r\cdot m$. For $r>0$,

$$\sum_{i=1}^{2}E_{i,2}(C_{i,2}(X))=X=D^{2^\wedge r\cdot m}(X)=\sum_{i=1}^{2}E_{i,2}(D^{2^\wedge(r-1)\cdot m}(C_{i,2}(X))).$$

Thus, $D^{2^\wedge(r-1)\cdot m}(C_{1,2}(X))=C_{1,2}(X)$; i.e., $D^{2^\wedge(r-1)\cdot m}(A)=A$. This implies that A is in a cycle with period less than or equal to $2^{r-1}\cdot m$. This contradiction shows that the period of the cycle containing X is $2\cdot(2^r\cdot m)=2^{r+1}\cdot m$ when $r>0$. On the other hand, if $r=0$, then

$$C_{1,2}(D^{m-1}(X))=D^{\frac{m-1}{2}}(C_{1,2}(X))=D^{\frac{m-1}{2}}(A)$$

and

$$C_{2,2}(D^{m-1}(X)) = D^{\frac{m-1}{2}}(C_{2,2}(X)) = (0, 0, \ldots, 0).$$

Since

$$C_{1,2}(D^m(X)) = C_{1,2}(D^{m-1}(X)) + C_{2,2}(D^{m-1}(X)) = D^{\frac{m-1}{2}}(A) \neq A = C_{1,2}(X),$$

we see that $D^m(X) \neq X$. Hence the period of the cycle containing X is $2 \cdot m$ when $r = 0$. Therefore there are cycles of $(2^s \cdot q)$-tuples with period $2^t \cdot m$ for $1 \leq t \leq s$.

We now show that there is a cycle of $(2^s \cdot q)$-tuples with period m. Suppose there is a $(2^{s-1} \cdot q)$-tuple B that is contained in a cycle of period m and for which each $C_{i,2^\wedge(s-1)}(B)$, $i = 1, \ldots, 2^{s-1}$, is also contained in a cycle of period m. By hypothesis, this holds for $s = 1$. Consider the $(2^s \cdot q)$-tuple

$$Y = E_{1,2}(B) + E_{2,2}(B + D^{\frac{m+1}{2}}(B)). \tag{21}$$

Now $C_{1,2}(Y) = B$ and $C_{2,2}(Y) = B + D^{\frac{m+1}{2}}(B)$; $C_{2,2}(Y)$ is also in a cycle of period m. Thus Y is in a cycle. We want to use Corollary 3 to show that the period of the cycle containing Y is m. Note that

$$\begin{cases} C_{i,2^\wedge s}(Y) = C_{\frac{i+1}{2},2^\wedge(s-1)}(B) & \text{when } i \text{ is odd,} \\ C_{i,2^\wedge s}(Y) = C_{\frac{i}{2},2^\wedge(s-1)}(B + D^{\frac{m+1}{2}}(B)) & \text{when } i \text{ is even.} \end{cases}$$

By assumption, when i is odd, the period of the cycle containing $C_{i,2^\wedge s}(Y)$ is m. To show that this is also the case when i is even, it suffices to show that the period of the cycle containing $C_{j,2^\wedge(s-1)}(B + D^{\frac{m+1}{2}}(B))$ is m for $j = 1, \ldots, 2^{s-1}$. Since $\gcd(m, 2^{s-1}) = 1$, there exist integers g and h for which

$$g \cdot m + h \cdot 2^{s-1} = \frac{m+1}{2}.$$

Either g or h is positive, but not both. Suppose $g > 0$ and $h < 0$. Then

$$B = D^{g \cdot m}(B) = D^{-h \cdot 2^\wedge(s-1)}(D^{\frac{m+1}{2}}(B)),$$

which implies

$$C_{j,2^\wedge(s-1)}(B) = D^{-h}(C_{j,2^\wedge(s-1)}(D^{\frac{m+1}{2}}(B))).$$

Hence, $C_{j,2^\wedge(s-1)}(D^{\frac{m+1}{2}}(B))$ is in the same cycle as $C_{j,2^\wedge(s-1)}(B)$. Since this cycle has period m, the cycle containing $C_{j,2^\wedge(s-1)}(D^{\frac{m+1}{2}}(B))$ also has period m. In a similar manner, it can be shown that this is also the case when $g < 0$ and $h > 0$. Since the cycle containing $C_{i,2^\wedge s}(Y)$, $i = 1, \ldots, 2^s$, has period m and since (21) holds, Corollary 3 implies that the cycle containing Y has period m. □

For a given n, the maximal period of cycles of Ducci-sequences is denoted by $P(n)$. By Corollary 3, if $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$, then $P(n)$ divides $2^s \cdot P(q)$. We now show that, in fact, $P(n) = 2^s \cdot P(q)$. This result appears in [2]; the proof there uses matrices and the fact that the cycle which has maximum period is generated by the n-tuple $(1, 0, \ldots, 0, 0)$. We offer a new proof here based on the compression functions. The result follows immediately from the proof of Theorem 7.

Theorem 8: Let $n = 2^s \cdot q$, where $s \geq 1$ and q is odd with $q > 1$. Then $P(n) = 2^s \cdot P(q)$.

Proof: Let A be a q-tuple that is contained in a cycle of period $P(q)$. Then the proof of Theorem 7 shows that the $(2^s \cdot q)$-tuple $X = E_{1,2^\wedge s}(A) = E_{1,2}(E_{1,2}(\ldots E_{1,2}(X)))$ is in a cycle of period $2^s \cdot P(q)$. □

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