ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-553 ***Proposed by Paul Bruckman, Berkeley, CA***

The following Diophantine equation has the trivial solution $(A, B, C, D) = (A, A, A, 0)$:

$$A^3 + B^3 + C^3 - 3ABC = D^k, \text{ where } k \text{ is a positive integer.} \tag{1}$$

Find nontrivial solutions of (1), i.e., with all quantities positive integers.

H-554 ***Proposed by N. Gauthier, Royal Military College of Canada***

Let k, a, and b be positive integers, with a and b relatively prime to each other, and define

$$\begin{aligned} N_k &:= (1+(-1)^k - L_k)^{-1} \\ &= (2-L_k)^{-1}, \ k \text{ even;} \\ &= -L_k^{-1}, \ k \text{ odd.} \end{aligned}$$

a. Show that

$$\begin{aligned} \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} L_{q(br+as)} = {} & N_{qa}N_{qb}[2 + L_{q(a+b)} - L_{qa} - L_{qb} - L_{qab} + (-1)^{qa} L_{qa(b-1)} \\ & + (-1)^{qb} L_{qb(a-1)} + (-1)^{q(a+b)+1} L_{q(ab-a-b)}] \\ & + N_q[(-1)^q L_{q(ab-1)} - L_{qab}], \end{aligned}$$

where q is a positive integer.

b. Show that

$$\begin{aligned} \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} F_{q(br+as)} = {} & N_{qa}N_{qb}[(-1)^{q(a+b)+1} F_{q(ab-a-b)} + F_{qa} + F_{qb} \\ & -F_{qab} + (-1)^{qa} F_{qa(b-1)} + F_{qb(a-1)} - F_{q(a+b)}] \\ & + N_q[(-1)^q F_{q(ab-1)} - F_{qab}], \end{aligned}$$

where q is a positive integer.

H-555 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Prove the following identity:

$$(x^n+y^n)(x+y)^n = -(-xy)^n + \sum_{k=0}^{[n/3]}(-1)^k C_{n,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n-3k}, \quad n=1,2,\ldots, \tag{1}$$

where

$$C_{n,k} = \binom{n-2k}{k}\cdot n/(n-2k).$$

Using (1), prove the following:

$$\text{(a)}\quad 5^{n/2}L_n = -1+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}5^k 4^{n-3k}, \quad n=2,4,6,\ldots; \tag{2}$$

$$\text{(b)}\quad 5^{(n+1)/2}F_n = 1+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}5^k 4^{n-3k}, \quad n=1,3,5,\ldots; \tag{3}$$

$$\text{(c)}\quad L_n = -1+\sum_{k=0}^{[n/3]}(-1)^k C_{n,k}2^{n-3k}, \quad n=1,2,3,\ldots. \tag{4}$$

SOLUTIONS

An Odd Problem

H-536 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 36, no. 1, February 1998)

Given an odd prime p, integers n and r with $n\geq 1$, let $m=2[\frac{1}{2}n]-1$,

$$S_{n,r,p}=\sum_{k=1}^{p-1}F_m^k\cdot\frac{F_{nk+r}}{k}, \quad T_{n,r,p}=\sum_{k=1}^{p-1}F_m^k\cdot\frac{L_{nk+r}}{k}.$$

Prove the following congruences:

(a) $S_{n,r,p}\equiv\dfrac{F_n^pF_{mp+r}-F_m^pF_{np+r}+F_r}{p} \pmod p$;

(b) $T_{n,r,p}\equiv\dfrac{F_n^pL_{mp+r}-F_m^pL_{np+r}+L_r}{p} \pmod p$.

Solution by the proposer

Proof: We begin with the following identity:

$$F_n\alpha^m = F_m\alpha^n - 1. \tag{$*$}$$

We may verify $(*)$ by dealing with the cases n even or n odd separately, then expanding the Binet formulas. A similar identity holds with the α's replaced by β's.

Raising each side of $(*)$ to the power p, we obtain:

$$F_n^p\alpha^{mp} = F_m^p\alpha^{np}-1+\sum_{k=1}^{p-1}\binom{p}{k}(-1)^{p-k}F_m^k\alpha^{nk} = F_m^p\alpha^{np}-1+\sum_{k=1}^{p-1}\binom{p-1}{k-1}\frac{p}{k}(-1)^{k-1}F_m^k\alpha^{nk}.$$

For $1 \le k \le p-1$,

$$\binom{p-1}{k-1} \equiv \binom{-1}{k-1} \pmod{p}$$
$$= (-1)^{k-1}.$$

Then, multiplying throughout by α^r, we obtain:

$$F_n^p \alpha^{mp+r} - F_m^p \alpha^{np+r} + \alpha^r \equiv p \sum_{k=1}^{p-1} F_m^k \frac{\alpha^{nk+r}}{k} \pmod{p^2}.$$

Note that the quantities "$1/k$" are the uniquely determined inverses k^{-1} (mod p^2); upon division throughout by p, these become the uniquely determined inverses k^{-1} (mod p). A similar congruence holds with the α's replaced by β's. Subtracting these two congruences and dividing throughout by $p\sqrt{5}$ yields the result in (a). Adding these two congruences and dividing throughout by p yields (b).

Note: Using these results, it may be shown that a necessary and sufficient condition for $Z(p^2) = Z(p)$ is that

$$S_{1,1,p} = \sum_{k=1}^{p-1} \frac{F_{k+1}}{k} \equiv 0 \pmod{p}.$$

Also solved by H.-J. Seiffert

A Recurrent Theme

H-537 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 36, no. 1, February 1998)***

Let $\langle w_n \rangle$ be any sequence satisfying the recurrence

$$w_{n+2} = Pw_{n+1} - Qw_n.$$

Let $e = w_0 w_2 - w_1^2$ and assume $e \ne 0$ and $Q \ne 0$.

Computer experiments suggest the following formula, where k is an integer larger than 1:

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_{k-i} \binom{k}{i} (-1)^i w_n^i w_{n+1}^{k-i},$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}.$$

Prove or disprove this conjecture.

Solution by Paul S. Bruckman, Berkeley, CA

We may express the w_n's in terms of the "fundamental" sequence $\langle \phi_n \rangle$, defined as follows:

$$\phi_n = (u^n - v^n)/(u-v), \tag{1}$$

where

$$u = \tfrac{1}{2}(P+\theta), \quad v = \tfrac{1}{2}(P-\theta), \quad \theta = (P^2 - 4Q)^{\frac{1}{2}}. \tag{2}$$

Note that $u+v=P$, $u-v=\theta$, and $uv=Q$. Also note that the ϕ_n's satisfy the same recurrence relation as the w_n,s, but have the initial values:

$$\phi_0=0, \quad \phi_1=1. \tag{3}$$

Also, $\phi_{-1}=-1/Q$, $\phi_2=P$. The formula for w_n is then as follows:

$$w_n=w_1\phi_n-Qw_0\phi_{n-1}. \tag{4}$$

We proceed to obtain closed form expressions for the indicated sums. First, we obtain a closed formula for the c_i's, substituting the expressions in (4):

$$\begin{aligned} c_i &= \theta^{-1}(uw_1-Qw_0)u^{i-1}\sum_{j=0}^{k-2}{}_{k-2}Cj(w_1)^{k-2-j}(-Qw_0/u)^j \\ &\quad -\theta^{-1}(vw_1-Qw_0)v^{i-1}\sum_{j=0}^{k-2}{}_{k-2}Cj(w_1)^{k-2-j}(-Qw_0/v)^j \\ &= \theta^{-1}(uw_1-Qw_0)u^{i-1}(w_1-vw_0)^{k-2}-\theta^{-1}(vw_1-Qw_0)v^{i-1}(w_1-uw_0)^{k-2} \end{aligned}$$

or

$$ci=\theta^{-1}u^i(w_1-vw_0)^{k-1}-\theta^{-1}v^i(w_1-uw_0)^{k-1}. \tag{5}$$

Next, let

$$S_{n,k}=\sum_{i=0}^{k}{}_kC_i(w_n)^{k-i}(-w_{n+1})^i c_{k-i}.$$

Note that this last expression differs from the sum given in the statement of the problem (with the roles of w_n and w_{n+1} interchanged). Substituting the expression in (5) yields:

$$S_{n,k}=\theta^{-1}\sum_{i=0}^{k}{}_kC_i(w_n)^{k-i}(-w_{n+1})^i\{u^{k-i}(w_1-vw_0)^{k-1}-v^{k-i}(w_1-uw_0)^{k-1}\}$$

or

$$S_{n,k}=\theta^{-1}(w_1-vw_0)^{k-1}(uw_n-w_{n+1})^k-\theta^{-1}(w_1-uw_0)^{k-1}(vw_n-w_{n+1})^k. \tag{6}$$

The problem (as corrected) asks us to verify or refute the relation

$$S_{n,k}=e^{k-1}w_{kn}. \tag{7}$$

Next, we employ the following relations [easily verified from the preceding relations, including (4)]:

$$uw_n-w_{n+1}=(uw_0-w_1)v^n, \tag{8}$$

$$vw_n-w_{n+1}=(vw_0-w_1)u^n. \tag{9}$$

It is also easily verified that

$$(uw_0-w_1)(vw_0-w_1)=-e. \tag{10}$$

Putting these facts together, we obtain (after simplification):

$$\begin{aligned} S_{n,k} &= \theta^{-1}(w_1-vw_0)^{k-1}(uw_n-w_{n+1})^k-\theta^{-1}(w_1-uw_0)^{k-1}(vw_n-w_{n+1})^k \\ &= e^{k-1}(w_1\phi_{kn}-Qw_0\phi_{kn-1})=e^{k-1}w_{kn}. \quad \text{Q.E.D.} \end{aligned}$$

Thus, there is a typographical error in the statement of the problem; the result is true only if the quantities w_n and w_{n+1} occurring in the first sum given are interchanged.

Also solved by H.-J. Seiffert

An Elementary Result

H-538 ***Proposed by Paul S. Bruckman, Highwood, IL***
(Vol. 36, no. 1, February 1998)

Define the sequence of integers $(B_k)_{k\ge 0}$ by the generating function:

$$(1-x)^{-1}(1+x)^{-\frac{1}{2}} = \sum_{k\ge 0} B_k \frac{\left(\frac{1}{2}x\right)^k}{k!}, \quad |x|<1 \text{ (see [1])}.$$

Show that

$$\sum_{k\ge 0} B_k^2 \cdot \frac{1}{(2k+2)!} = \frac{\pi^2}{8} - \frac{1}{4}\log^2 u, \text{ where } u = 1+\sqrt{2}.$$

Reference

1. P. S. Bruckman. "An Interesting Sequence of Numbers Derived from Various Generating Functions." *The Fibonacci Quarterly* **10.2** (1972):169-81.

Solution by the proposer

In [1], it is shown that

$$\tan^{-1} x \cdot (1-x^2)^{-1/2} = \sum_{k\ge 0} B_k^2 \frac{x^{2k+1}}{(2k+1)!}.$$

The following result is Elementary Problem E3140, Part (b)(ii), proposed by Khristo Boyadzhiev in *The American Math Monthly* **93.3** (1986):216:

$$\int_0^1 \tan^{-1} x \cdot (1-x^2)^{-1/2} dx = \pi^2/8 - \tfrac{1}{4}\log^2 u.$$

(The notation is modified to conform to our own.) The result follows immediately, by integrating the series given in [1] term by term and evaluating it at the integral's limits.

Beta Version

H-539 ***Proposed by H.-J. Seiffert, Berlin, Germany***
(Vol. 36, no. 2, May 1998)

Let

$$H_m(p) = \sum_{j=1}^{m} B\left(\frac{j}{2}, p\right), m \in N, \ p>0,$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

denotes the Beta function. Show that for all positive reals p and all positive integers n,

$$\sum_{k=1}^{n} (-1)^{k-1}\binom{n}{k} H_{2k}(p) = 4^{n+p-1}B(n+p, n+p-1) + \frac{1}{n+p-1}. \tag{1}$$

From (1), deduce the identities

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{4^k}\binom{n}{k}\binom{2k}{k}=\frac{2}{4^n}\binom{2n-2}{n-1} \tag{2}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}4^k\binom{n}{k}\Big/\binom{2k}{k}=\frac{2n}{2n-1}. \tag{3}$$

Solution by the proposer

Since

$$B\left(\frac{j}{2},p\right)=\int_0^1 t^{(j-2)/2}(1-t)^{p-1}dt=2\int_0^1 u^{j-1}(1-u^2)^{p-1}du,$$

it easily follows that

$$H_m(p)=2\int_0^1\frac{1-u^m}{1-u}(1-u^2)^{p-1}du,\ m\in N.$$

If $S_n(p)$ denotes the left side of the stated identity (1), then, by the Binomial theorem,

$$S_n(p)=2\int_0^1\left(\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(1-u^{2k})\right)\frac{(1-u^2)^{p-1}}{1-u}du$$

$$=2\int_0^1\left(\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}u^{2k}\right)\frac{(1-u^2)^{p-1}}{1-u}du=2\int_0^1(1-u^2)^n\frac{(1-u^2)^{p-1}}{1-u}du$$

or

$$\frac{1}{2}S_n(p)=\int_0^1(1-u)^{n+p-2}(1+u)^{n+p-1}du.$$

Substituting $u=1-2v$ yields

$$\frac{1}{2}S_n(p)=4^{n+p-1}\int_0^{1/2}v^{n+p-2}(1-v)^{n+p-1}dv. \tag{4}$$

Integrating (4) by parts, we find

$$\frac{1}{2}S_n(p)=\frac{1}{n+p-1}+4^{n+p-1}\int_0^{1/2}v^{n+p-1}(1-v)^{n+p-2}dv.$$

Replacing v by $1-v$ in the latter integral, we get

$$\frac{1}{2}S_n(p)=\frac{1}{n+p-1}+4^{n+p-1}\int_{1/2}^{1}v^{n+p-2}(1-v)^{n+p-1}dv. \tag{5}$$

Now, the desired identity (1) follows by adding (4) and (5).

Interestingly, (2) and (3) will follow from (1), simultaneously, when taking $p=1/2$. Since, as is well known,

$$B\left(r+\frac{1}{2},\frac{1}{2}\right)=\frac{\pi}{4^r}\binom{2r}{r},\ r\in N_0,\ \text{ and }\ B\left(r,\frac{1}{2}\right)=\frac{4^r}{r}\Big/\binom{2r}{r},\ r\in N,$$

we have

$$H_{2k}\left(\frac{1}{2}\right)=\sum_{r=0}^{k-1}B\left(r+\frac{1}{2},\frac{1}{2}\right)+\sum_{r=1}^{k}B\left(r,\frac{1}{2}\right)=\pi\sum_{r=0}^{k-1}\frac{1}{4^r}\binom{2r}{r}+\sum_{r=1}^{k}\frac{4^r}{r}\Big/\binom{2r}{r}.$$

Each of the equations

$$\sum_{r=0}^{k-1}\frac{1}{4^r}\binom{2r}{r}=\frac{2k}{4^k}\binom{2k}{k},\ k\in N,\quad\text{and}\quad\sum_{r=1}^{k}\frac{4^r}{r}\Big/\binom{2r}{r}=2\left(4^k\Big/\binom{2k}{k}-1\right),\ k\in N,$$

can be proved by a simple induction argument. Hence,

$$H_{2k}\left(\frac{1}{2}\right)=\frac{2k}{4^k}\binom{2k}{k}\pi+2\left(4^k\Big/\binom{2k}{k}-1\right),\ k\in N.$$

Using

$$B\left(n+\frac{1}{2},n-\frac{1}{2}\right)=\frac{2\pi}{4^{2n-1}}\binom{2n-2}{n-1}$$

and observing that π is an irrational number, from (1) with $p=1/2$, we find the two equations

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{2k}{4^k}\binom{n}{k}\binom{2k}{k}=\frac{1}{4^{n-1}}\binom{2n-2}{n-1} \tag{6}$$

and

$$2\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\left(4^k\Big/\binom{2k}{k}-1\right)=\frac{2}{2n-1}. \tag{7}$$

Obviously, (2) is equivalent to (6). Dividing (7) by 2 and adding

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}=1$$

to both sides of the resulting equation gives (3).

With $p=1$, identity (1), after dividing by 2, gives

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_{2k}=2^{2n-1}\frac{n!(n-1)!}{(2n)!}+\frac{1}{2n},$$

where $H_m=H_m(1)/2=\sum_{j=1}^{m}1/j$ is the m^{th} harmonic number. This equation (including a generalization in another direction) was obtained in [1].

Reference

1. L. C. Hsu & H. Kappus. Problem B-818. *The Fibonacci Quarterly* **35.3** (1997):280-81.

Also solved by P. Bruckman and partially by A. Stam.

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