SUMS OF CERTAIN PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

Inspired by the charming result

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \tag{1.1}$$

Clary and Hemenway [3] discovered factored closed-form expressions for all sums of the form $\sum_{k=1}^{n} F_{rk}^3$, where r is an integer. One of their main aims was to find sums that could be expressed neatly as products of Fibonacci and Lucas numbers. At the end of their paper they mentioned the result

$$\sum_{k=1}^{n} F_k^2 F_{k+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}, \tag{1.2}$$

published by Block [2] in 1953.

Motivated by (1.1) and (1.2), we have discovered an infinity of similar identities which we believe are new. For example, we have found

$$\sum_{k=1}^{n} F_k F_{k+1} F_{k+2}^2 F_{k+3} F_{k+4} = \frac{1}{4} F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}, \tag{1.3}$$

and

$$\sum_{k=1}^{n} F_k F_{k+1} F_{k+2} F_{k+3} F_{k+4}^2 F_{k+5} F_{k+6} F_{k+7} F_{k+8} = \frac{1}{11} F_n F_{n+1} \dots F_{n+9}. \tag{1.4}$$

In Section 2 we prove a theorem involving a sum of products of Fibonacci numbers, and in Section 3 we prove the corresponding theorem for the Lucas numbers. In Section 4 we present three additional theorems, two of which involve sums of products of squares of Fibonacci and Lucas numbers.

We require the following identities:

$$F_{n+k} + F_{n-k} = F_n L_k, \ k \text{ even}, \tag{1.5}$$
$$F_{n+k} + F_{n-k} = L_n F_k, \ k \text{ odd}, \tag{1.6}$$
$$F_{n+k} - F_{n-k} = F_n L_k, \ k \text{ odd}, \tag{1.7}$$
$$F_{n+k} - F_{n-k} = L_n F_k, \ k \text{ even}, \tag{1.8}$$
$$L_{n+k} + L_{n-k} = L_n L_k, \ k \text{ even}, \tag{1.9}$$
$$L_{n+k} + L_{n-k} = 5F_n F_k, \ k \text{ odd}, \tag{1.10}$$
$$L_{n+k} - L_{n-k} = L_n L_k, \ k \text{ odd}, \tag{1.11}$$
$$L_{n+k} - L_{n-k} = 5F_n F_k, \ k \text{ even}, \tag{1.12}$$
$$L_n^2 - L_{2n} = -2 = -L_0, \ n \text{ odd}, \tag{1.13}$$

$$5F_{2n}^2 - L_{2n}^2 = -4 = -L_0^2, \tag{1.14}$$
$$5F_{2n}^2 - L_{4n} = -2 = -L_0. \tag{1.15}$$

Identities (1.5)-(1.8) occur on page 59 of Hoggatt [4], while (1.9)-(1.12) occur as (9)-(12), respectively, in Bergum and Hoggatt [1]. Identities (1.13)-(1.15) can be proved with the use of the Binet forms.

2. A FAMILY OF SUMS FOR THE FIBONACCI NUMBERS

Theorem 1: Let m be a positive integer. Then

$$\sum_{k=1}^{n} F_k F_{k+1} \ldots F_{k+2m}^2 \ldots F_{k+4m} = \frac{F_n F_{n+1} \ldots F_{n+4m+1}}{L_{2m+1}}. \tag{2.1}$$

Proof: We use the elegant method described on page 135 in [3] to prove (1.2). Let l_n and r_n denote the left and right sides, respectively, of (2.1). Then $l_n - l_{n-1} = F_n F_{n+1} \ldots F_{n+2m}^2 \ldots F_{n+4m}$. Also,

$$\begin{aligned} r_n - r_{n-1} &= \frac{F_n F_{n+1} \ldots F_{n+4m}}{L_{2m+1}} [F_{n+4m+1} - F_{n-1}] \\ &= \frac{F_n F_{n+1} \ldots F_{n+4m}}{L_{2m+1}} [F_{(n+2m)+(2m+1)} - F_{(n+2m)-(2m+1)}] \\ &= l_n - l_{n-1} \quad \text{using (1.7)}. \end{aligned}$$

Hence, to prove that $l_n = r_n$ it suffices to show that $l_1 = r_1$. But

$$\begin{aligned} r_1 &= \frac{F_1 F_2 \ldots F_{4m+1} F_{2m+1} L_{2m+1}}{L_{2m+1}} \quad (\text{since } F_{2n} = F_n L_n) \\ &= l_1, \quad \text{and this completes the proof. } \square \end{aligned}$$

When $m = 1$ and 2, identity (2.1) reduces to (1.3) and (1.4), respectively. However, while (1.1) and (1.2) can be proved in a similar way, they are not special cases of (2.1).

3. CORRESPONDING RESULTS FOR THE LUCAS NUMBERS

Corresponding to (1.1) we have

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2, \tag{3.1}$$

which occurs as I_4 in Hoggatt [4]. The Lucas counterpart to (1.2) is

$$\sum_{k=1}^{n} L_k^2 L_{k+1} = \frac{1}{2} L_n L_{n+1} L_{n+2} - 3. \tag{3.2}$$

The constants on the right sides of (3.1) and (3.2) can be obtained by trial, and also in the same manner as in our next theorem, demonstrating a certain unity.

Theorem 2: Let m be a positive integer. Then

$$\sum_{k=1}^{n} L_k L_{k+1} \ldots L_{k+2m}^2 \ldots L_{k+4m} = \frac{L_n L_{n+1} \ldots L_{n+4m+1}}{L_{2m+1}} - R_0, \tag{3.3}$$

where

$$R_n = \frac{L_n L_{n+1} \cdots L_{n+4m+1}}{L_{2m+1}}, \quad n = 0, 1, 2, \ldots.$$

Proof: Again, let l_n denote the left side of (3.3). Then

$$\begin{aligned}
R_n - R_{n-1} &= \frac{L_n L_{n+1} \cdots L_{n+4m}}{L_{2m+1}} [L_{n+4m+1} - L_{n-1}] \\
&= \frac{L_n L_{n+1} \cdots L_{n+4m}}{L_{2m+1}} [L_{(n+2m)+(2m+1)} - L_{(n+2m)-(2m+1)}] \\
&= L_n L_{n+1} \cdots L_{n+2m}^2 \cdots L_{n+4m} \quad \text{[by (1.11)]} \\
&= l_n - l_{n-1}.
\end{aligned}$$

From this we see that $l_n - R_n = c$, where c is a constant. Now,

$$\begin{aligned}
c &= l_1 - R_1 \\
&= L_1 L_2 \ldots L_{4m+1} \left[L_{2m+1} - \frac{L_{4m+2}}{L_{2m+1}} \right] \\
&= L_1 L_2 \ldots L_{4m+1} \cdot \frac{L_{2m+1}^2 - L_{4m+2}}{L_{2m+1}} \\
&= -\frac{L_0 L_1 L_2 \ldots L_{4m+1}}{L_{2m+1}} \quad \text{[by (1.13)]} \\
&= -R_0.
\end{aligned}$$

This concludes the proof. □

Since this method of proof applies to (3.1) and (3.2), we see that the appropriate constants on the right sides are $-2 = -L_0 L_1$ and $-3 = -\frac{1}{2} L_0 L_1 L_2$, respectively. Accordingly, we write (3.1), for example, as

$$\sum_{k=0}^{n} L_k^2 = [L_k L_{k+1}]_0^n.$$

We use this notation throughout the remainder of the paper.

Remark: If for $m = 0$ we interpret the summands in (2.1) and (3.3) to be F_k^2 and L_k^2, respectively, then we can realize (1.1) and (3.1) within the framework of our two theorems. However, the same cannot be said for (1.2) and (3.2).

4. MORE SUMS OF PRODUCTS

In this section we state three additional theorems, two of which involve sums of products of squares. Using (1.5)-(1.15), they can be proved in the same manner as Theorems 1 and 2, and so we leave this task to the reader. In each theorem, m is assumed to be a nonnegative integer.

Theorem 3:

$$\sum_{k=1}^{n} F_k F_{k+1} \ldots F_{k+4m+2} L_{k+2m+1} = \frac{F_n F_{n+1} \ldots F_{n+4m+3}}{F_{2m+2}}, \tag{4.1}$$

$$\sum_{k=1}^{n} L_k L_{k+1} \dots L_{k+4m+2} F_{k+2m+1} = \left[\frac{L_k L_{k+1} \dots L_{k+4m+3}}{5F_{2m+2}} \right]_0^n. \tag{4.2}$$

Theorem 4:

$$\sum_{k=1}^{n} F_k^2 F_{k+1}^2 \dots F_{k+4m}^2 F_{2k+4m} = \frac{F_n^2 F_{n+1}^2 \dots F_{n+4m+1}^2}{F_{4m+2}}, \tag{4.3}$$

$$\sum_{k=1}^{n} L_k^2 L_{k+1}^2 \dots L_{k+4m}^2 F_{2k+4m} = \left[\frac{L_k^2 L_{k+1}^2 \dots L_{k+4m+1}^2}{5F_{4m+2}} \right]_0^n. \tag{4.4}$$

In the proof of (4.3), when finding $r_n - r_{n-1}$, we obtain the expression $F_{n+4m+1}^2 - F_{n-1}^2$, which by (1.6) and (1.7) can be written as

$$\begin{aligned}[F_{(n+2m)+(2m+1)} - F_{(n+2m)-(2m+1)}][F_{(n+2m)+(2m+1)} + F_{(n+2m)-(2m+1)}] \\ = F_{n+2m}L_{2m+1} \cdot L_{n+2m}F_{2m+1} = F_{2n+4m}F_{4m+2}.\end{aligned}$$

Similar expressions that arise in the proof of (4.4), and in the proof of the next theorem, can be treated in the same manner.

A simple special case of (4.3), which occurs for $m = 0$, is $\sum_{k=1}^{n} F_k^2 F_{2k} = F_n^2 F_{n+1}^2$.

Theorem 5:

$$\sum_{k=1}^{n} F_k^2 F_{k+1}^2 \dots F_{k+4m+2}^2 F_{2k+4m+2} = \frac{F_n^2 F_{n+1}^2 \dots F_{n+4m+3}^2}{F_{4m+4}}, \tag{4.5}$$

$$\sum_{k=1}^{n} L_k^2 L_{k+1}^2 \dots L_{k+4m+2}^2 F_{2k+4m+2} = \left[\frac{L_k^2 L_{k+1}^2 \dots L_{k+4m+3}^2}{5F_{4m+4}} \right]_0^n. \tag{4.6}$$

To conclude we mention that, for p real, the sequences $\{U_n\}$ and $\{V_n\}$, defined for all integers n by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2,\ V_1 = p, \end{cases}$$

generalize the Fibonacci and Lucas numbers, respectively. The results contained in Theorems 1-5 translate immediately to U_n and V_n. The reason is that if we replace F_n by U_n, L_n by V_n, and 5 by $p^2 + 4$, then U_n and V_n satisfy (1.5)-(1.15).

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