ON ∞-GENERALIZED FIBONACCI SEQUENCES

Walter Motta
Departamento de Matemática, CETEC-UFU,
Campus Santa Mônica, 38400-902 Uberlândia, MG, Brazil
e-mail: wmotta@ufu.br

Mustapha Rachidi
Départment de Mathématiques, Faculté des Sciences,
Université Mohammed V, B.P. 1014, Rabat, Morocco
e-mail: rachidi@fsr.ac.ma

Osamu Saeki
Department of Mathematics, Faculty of Science,
Hiroshima University, Higsashi-Hiroshima 739-8526, Japan
e-mail: saeki@math.sci.hiroshima-u.ac.jp

1. INTRODUCTION

Let $a_0, a_1, \ldots, a_{r-1}$ be arbitrary complex numbers with $a_{r-1} \neq 0$ $(1 \leq r < \infty)$. For a given sequence of complex numbers $A = (\alpha_{-r+1}, \alpha_{-r+2}, \ldots, \alpha_0)$, we define the *weighted r-generalized Fibonacci sequence* $\{y_A(n)\}_{n=-r+1}^{\infty}$ by using a recurrence formula involving $r+1$ terms* as follows:

$$y_A(n) = \alpha_n \quad (n = -r+1, -r+2, \ldots, 0);$$

$$y_A(n) = \sum_{i=1}^{r} a_{i-1} y_A(n-i) \quad (n = 1, 2, 3, \ldots).$$

When $a_i = 1$ for all i and $A = (0, 0, \ldots, 0, 1)$, we get the r-generalized Fibonacci numbers (see [4]). A Binet-type formula and a combinatorial expression of weighted r-generalized Fibonacci sequences are given in [3]. Furthermore, in [2], the convergence of the sequence $\{y_A(n) / n^{\nu-1} q^n\}$ has been studied, where q is a root of the characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \cdots - a_{r-2} x - a_{r-1}$ of multiplicity ν.

The purpose of this paper is to generalize the weighted r-generalized Fibonacci sequences with $1 \leq r < \infty$ to a class of sequences which are defined by recurrence formulas involving *infinitely many* terms, and to analyze their asymptotic behavior. We call such sequences *∞-generalized Fibonacci sequences*. This is a new generalization of the usual Fibonacci sequences and almost nothing has been known about such sequences until now. For example, there has been no theory of difference equations for such sequences.

More precisely, an ∞-generalized Fibonacci sequence is defined as follows. We suppose that two infinite sequences of complex numbers are given, one for the initial sequence and the other for the weight sequence. Then a member of the ∞-generalized Fibonacci sequence is determined by the weighted series of its preceding members (for a precise definition, see §2). Since the recurrence formula always involves infinitely many terms, we always have to worry about the convergence of the series corresponding to the recurrence formula and hence we need auxiliary conditions on the initial sequence and the weight sequence.

* This is called an *r-th order linear recurrence* in [3].

One of the striking results of this paper is that, under certain conditions, an ∞-generalized Fibonacci sequence behaves very much like a weighted r-generalized Fibonacci sequence with r finite, as far as its asymptotic behavior is concerned.

The paper is organized as follows. In §2 we give a precise definition of the ∞-generalized Fibonacci sequences. In §3 we analyze their asymptotic behavior under certain conditions. In §4 we give some explicit examples in order to illustrate our results.

2. ∞-GENERALIZED FIBONACCI SEQUENCES

Take an infinite sequence $\{a_i\}_{i=0}^{\infty}$ of complex numbers, which will later be the weight sequence of ∞-generalized Fibonacci sequences. We set $h(z)=\sum_{i=0}^{\infty}a_iz^i$ for $z\in\mathbf{C}$ and $u(x)=\sum_{i=1}^{\infty}|a_i|x^i$ for $x\in\mathbf{R}$. Let R denote the radius of convergence of the power series h, which coincides with the radius of convergence of u. We assume the following condition:

$$0<R\leq\infty. \tag{2.1}$$

Let X be the set of sequences $\{x_i\}_{i=0}^{\infty}$ of complex numbers such that there exist $C>0$ and T with $0<T<R$ satisfying $|x_i|\leq CT^i$ for all i. Note that X is an infinite dimensional vector space over $\mathbf{C}$; it will be the set of initial sequences for ∞-generalized Fibonacci sequences associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Define $f:X\to\mathbf{C}$ by $f(x_0,x_1,\dots)=\sum_{i=0}^{\infty}a_ix_i$. Since the series $\sum_{i=0}^{\infty}a_iCT^i$ converges absolutely, the series defining f also converges absolutely.

Lemma 2.2: If $\{y_0,y_{-1},y_{-2},\dots\}\in X$, then the sequence $\{y_m,y_{m-1},y_{m-2},\dots,y_1,y_0,y_{-1},y_{-2},\dots\}$ is an element of X for every finite sequence of complex numbers $y_m,y_{m-1},\dots,y_1$ $(m\geq 1)$.

Proof: By our assumption, there exist $C>0$ and T with $0<T<R$ such that $|y_{-i}|\leq CT^i$ for all $i\geq 0$. Then we have $|y_{-i}|\leq(CT^{-m})T^{i+m}$ for all $i\geq 0$. On the other hand, there exists $C'>0$ such that $|y_{m-j}|\leq C'T^j$ for $j=0,1,\dots,m-1$. Putting $C''=\max\{C',CT^{-m}\}$, we have $|y_{m-j}|\leq C''T^j$ for all $j\geq 0$. This completes the proof. □

Now we define an ∞-generalized Fibonacci sequence as follows. For a sequence $\{y_0,y_{-1},y_{-2},\dots\}\in X$, we define the sequence $\{y_1,y_2,y_3,\dots\}$ by

$$y_n=f(y_{n-1},y_{n-2},y_{n-3},\dots)=\sum_{i=1}^{\infty}a_{i-1}y_{n-i}\quad(n=1,2,3,\dots).$$

This is well defined by Lemma 2.2. The sequence $\{y_i\}_{i\in\mathbf{Z}}$ is called an *∞-generalized Fibonacci sequence* associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Note that if there exists an integer $r\geq 1$ such that $a_i=0$ for all $i\geq r$, then the sequence $\{a_i\}_{i=0}^{\infty}$ satisfies the condition (2.1) and the above definition coincides with that of weighted r-generalized Fibonacci sequences. Thus ∞-generalized sequences generalize weighted r-generalized Fibonacci sequences with r finite.

Lemma 2.3:

(1) Suppose that each a_i is a nonnegative real number and that there exists an S with $0<S<R$ satisfying

$$a_0>S^{-1}-u(S)\ \text{ (or, equivalently, } Sh(S)>1). \tag{2.3.1}$$

Then there exists a unique $q\in\mathbf{R}$ such that $q>S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty}\in X$, and $f(q^{-1},q^{-2},q^{-3},\dots)=1$.

(2) Suppose that there exists an S with $0 < S < R$ satisfying

$$|a_0| > S^{-1} + u(S). \tag{2.3.2}$$

Then there exists a unique $q \in \mathbf{C}$ such that $|q| > S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$, and $f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$.

Proof:

(1) For $x > R^{-1}$, set $\varphi(x) = f(x^{-1}, x^{-2}, x^{-3}, \dots) = x^{-1}h(x^{-1})$. Note that φ is a differentiable function. Then we have $\lim_{x\to\infty}\varphi(x) = 0$ and $\varphi'(x) = f(-x^{-2}, -2x^{-3}, \dots) < 0$ for all $x > R^{-1}$. Furthermore, we have $\varphi(S^{-1}) > 1$ by (2.3.1). Then the intermediate value theorem implies that there exists a unique $q > S^{-1}$ such that $\varphi(q) = 1$.

(2) Define the holomorphic function $v(z)$ by $v(z) = 1 - \sum_{i=1}^{\infty} a_i z^{i+1}$ for z with $|z| < R$. Then, for z with $|z| = S$, we have

$$|v(z)| \le 1 + \sum_{i=1}^{\infty} |a_i| |z|^{i+1} = 1 + Su(S) \; < \; |a_0| S = |a_0 z|$$

by (2.3.2). Hence, by Rouché's theorem, $a_0 z - v(z)$ and $a_0 z$ have the same number of zeros in the region $|z| < S$. Note that $a_0 z - v(z) = 0$ if and only if $zh(z) = 1$. Since $a_0 z$ has a unique zero in the region, we have the conclusion. □

Remark 2.4: For a weighted r-generalized Fibonacci sequence of nonnegative real numbers with r finite, condition (2.3.1) is always satisfied, and the real number q as in Lemma 2.3(1) is the unique positive real root of the characteristic polynomial (not necessarily asymptotically simple in the terminology of [2]).

Remark 2.5: In the situation of the above lemma, if $\{y_0, y_{-1}, y_{-2}, \dots\} = \{1, q^{-1}, q^{-2}, q^{-3}, \dots\}$, then we can check easily that $y_n = q^n$ for all $n \in \mathbf{Z}$.

Note that if condition (2.3.1) or (2.3.2) is not satisfied, then, in general, there exists no $q \ne 0$ such that $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$ and $f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$. For example, consider $a_i = -1/(i+1)!$. The sequence $\{a_i\}_{i=0}^{\infty}$ satisfies condition (2.1) with $R = \infty$. However, $1 - zh(z) = e^z$ and there exists no $q \ne 0$ with $q^{-1}h(q^{-1}) = 1$.

3. CONVERGENCE RESULT FOR $\lim_{n\to\infty} y_n / q^n$

Our aim in this section is to prove a convergence theorem for the sequence $\{y_n / q^n\}$ (Theorem 3.10), where $\{y_n\}$ is an ∞-generalized Fibonacci sequence as defined in §2 and q is as in Lemma 2.3.

We first define the auxiliary sequence $\{g_n\}$ as follows. We set $g_0 = 1$, $g_n = 0$ for $n \le -1$, and define $\{g_n\}_{n=1}^{\infty}$ as the ∞-generalized Fibonacci sequence associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$ and the initial sequence $\{g_n\}_{n=0}^{-\infty}$; i.e., $g_1 = f(g_0, g_{-1}, g_{-2}, \dots)$, $g_2 = f(g_1, g_0, g_{-1}, \dots)$, etc.

Lemma 3.1: For all $n \ge 1$, we have

$$y_n = g_n y_0 + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} g_{n-j} a_{i+j-1}\right) y_{-i}.$$

Furthermore, the series on the right-hand side converges absolutely; i.e., the following series converges:

$$|g_n y_0| + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n}|g_{n-j}a_{i+j-1}|\right)|y_{-i}|.$$

Proof: Note that $g_1 = a_0$. Then the equality for $n=1$ together with the absolute convergence is easily checked. Now assume that, for $n, n-1, n-2, \dots, 1$, the right-hand side of the equality converges absolutely and that the equality is valid. Then we have

$$\begin{aligned}
y_{n+1} &= \sum_{i=0}^{\infty} a_i y_{n-i} \\
&= \sum_{i=0}^{n-1} a_i\left(g_{n-i}y_0 + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n-i} g_{n-i-j}a_{k+j-1}\right)y_{-k}\right) + \sum_{i=n}^{\infty} a_i y_{n-i} \\
&= \left(\sum_{i=0}^{n} a_i g_{n-i}\right)y_0 + \sum_{i=0}^{n-1} a_i \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n-i} g_{n-i-j}a_{k+j-1}\right)y_{-k} + \sum_{k=1}^{\infty} a_{n+k}y_{-k} \\
&= g_{n+1}y_0 + \sum_{k=1}^{\infty}\left(\sum_{i=0}^{n-1} a_i \sum_{j=1}^{n-i} g_{n-i-j}a_{k+j-1} + a_{n+k}\right)y_{-k} \\
&= g_{n+1}y_0 + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n}\left(\sum_{i=0}^{n-j} a_i g_{n-j-i}\right)a_{k+j-1} + g_0 a_{n+k}\right)y_{-k} \\
&= g_{n+1}y_0 + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n+1} g_{n+1-j}a_{k+j-1}\right)y_{-k}.
\end{aligned}$$

Note that we can change the order of addition, since each of the series appearing in the second line converges absolutely. Thus, the equality is valid also for $n+1$ and the right-hand side converges absolutely. □

Set

$$b_m = \sum_{i=m}^{\infty}\frac{a_i}{q^{i+1}} \quad (m \ge 0),$$

where q is as in Lemma 2.3. Note that $b_0 = f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$. By the previous lemma combined with Remark 2.5, we have, for $n \ge 1$,

$$q^n = g_n + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} g_{n-j}a_{i+j-1}\right)q^{-i}.$$

Hence, we have

$$1 = \frac{g_n}{q^n} + \sum_{i=1}^{\infty}\sum_{j=1}^{n}\frac{g_{n-j}}{q^{n-j}}\cdot\frac{a_{i+j-1}}{q^{i+j}} = \frac{g_n}{q^n} + \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty}\frac{a_{i+j-1}}{q^{i+j}}\right)\frac{g_{n-j}}{q^{n-j}} = \frac{g_n}{q^n} + \sum_{j=1}^{n} b_j\frac{g_{n-j}}{q^{n-j}}.$$

In other words, we have $1 = b_0c_n + b_1c_{n-1} + b_2c_{n-2} + \cdots + b_nc_0$ for all $n \ge 0$, where $c_n = g_n / q^n$. We will show that $\lim_{n\to\infty} c_n$ exists. Set $k_n = c_n - c_{n-1}$.

Lemma 3.2: For all $n \geq 1$, we have

$$k_n = \sum_{(i_1, \dots, i_s) \in \Theta_n} (-1)^s b_{i_1} \cdots b_{i_s},$$

where Θ_n is the finite set defined by

$$\Theta_n = \left\{ (i_1, \dots, i_s) : i_j \in \mathbf{Z},\ i_j \geq 1,\ s \geq 1, \sum_{j=1}^{s} i_j = n \right\}.$$

Proof: First, note that $k_0 b_0 = 1$ and that $k_0 b_n + k_1 b_{n-1} + \cdots + k_n b_0 = 0$ $(n \geq 1)$. The equality is easily checked for $n = 1$. Suppose that the equality is valid for $n, n-1, \dots, 1$. We put $\Theta_0 = \{\emptyset\}$ and adopt the convention that the sum over Θ_0 is equal to 1. Then we have

$$k_{n+1} = -b_1 k_n - b_2 k_{n-1} - \cdots - b_{n+1} k_0 = -\sum_{i=1}^{n+1} b_i \sum_{(i_1, \dots, i_r) \in \Theta_{n+1-i}} (-1)^r b_{i_1} \cdots b_{i_r}.$$

On the other hand, we have

$$\Theta_{n+1} = \bigcup_{i=1}^{n+1} \{(i, i_1, \dots, i_r) : (i_1, \dots, i_r) \in \Theta_{n+1-i}\}.$$

Then it follows that

$$k_{n+1} = \sum_{(i_1, \dots, i_r) \in \Theta_{n+1}} (-1)^r b_{i_1} \cdots b_{i_r}.$$

This completes the proof. □

Lemma 3.3: If $\sum_{m=1}^{\infty} |b_m| < 1$, then the series $\sum_{n=0}^{\infty} k_n$ converges absolutely and is equal to $(\sum_{m=0}^{\infty} b_m)^{-1}$.

Proof: First, note that the series $\sum_{i=0}^{\infty} (-1)^i z^i$ converges absolutely for $|z| < 1$ and is equal to $(1+z)^{-1}$. Since $\sum_{m=1}^{\infty} |b_m| < 1$ by our assumption, we see that the series $\sum_{i=0}^{\infty} (-1)^i (\sum_{m=1}^{\infty} b_m)^i$ converges absolutely and is equal to $(1 + \sum_{m=1}^{\infty} b_m)^{-1} = (\sum_{m=0}^{\infty} b_m)^i$. Hence, we can change the order of addition in the series $\sum_{i=0}^{\infty} (-1)^i (\sum_{m=1}^{\infty} b_m)^i$. Then, using Lemma 3.2, it is not hard to verify that, changing the order of addition appropriately, this series coincides with the series $\sum_{n=0}^{\infty} k_n$. This completes the proof. □

Note that Lemma 3.3 is an analog of Lemma 13 and Theorem 14 of [2]. However, the method in [2] cannot be applied directly to our case.

Proposition 3.4: Suppose that there exists an S with $0 < S < R$ satisfying (2.3.1) or (2.3.2), and

$$S^2 u'(S) < 1. \tag{3.4.1}$$

Then $\lim_{n \to \infty} c_n = \lim_{n \to \infty} g_n / q^n$ exists and is equal to $(1 + 1^{-2} h'(q^{-1}))^{-1} = (\sum_{m=0}^{\infty} b_m)^{-1}$.

Proof: Since $k_0 b_0 = 1$ and $k_0 b_n + k_1 b_{n-1} + \cdots + k_n b_0 = 0$ for all $n \geq 1$, we see that

$$\sum_{j=0}^{\infty} \sum_{i=0}^{j} k_i b_{j-i} = 1.$$

On the other hand, we have

$$\sum_{m=1}^{\infty}|b_m| = \sum_{m=1}^{\infty}\left|\sum_{i=m}^{\infty}\frac{a_i}{q^{i+1}}\right| \le \sum_{m=1}^{\infty}\sum_{i=m}^{\infty}|a_i|S^{i+1} = \sum_{i=1}^{\infty} i|a_i|S^{i+1} = S^2u'(S) < 1$$

by (3.4.1). Thus, $\lim_{n\to\infty} c_n = \Sigma_{n=0}^{\infty} k_n$ converges absolutely by Lemma 3.3. Therefore, we have $(\Sigma_{n=0}^{\infty} k_n)(\Sigma_{m=0}^{\infty} b_m) = 1$, since $\Sigma_{m=0}^{\infty} b_m$ converges absolutely. On the other hand, we have

$$q^{-2}h'(q^{-1}) = q^{-2}\left(\sum_{i=0}^{\infty} ia_i q^{-(i-1)}\right) = \sum_{i=1}^{\infty} ia_i q^{-(i+1)},$$

and $\Sigma_{m=0}^{\infty} b_m = 1 + \Sigma_{i=1}^{\infty} ia_i q^{-(i+1)}$. This completes the proof. □

Note that the limit as in Proposition 3.4 does not always exist in general as is seen in [2] if we drop the condition (3.4.1). When there exists an r with $a_i = 0$ $(i \ge r)$, the above lemma shows that the sequence is asymptotically simple with dominant root q and dominant multiplicity 1 in the terminology of [2].

Remark 3.5: Note that it is easy to construct sequences which satisfy condition (2.1) and which admit a real number S with $0 < S < R$ satisfying (2.3.1) or (2.3.2), and (3.4.1). For example, take an arbitrary holomorphic function $h_1(z)$ defined in a neighborhood of zero. Then the sequence appearing as the coefficients of the power series expansion of the holomorphic function $h(z) = h_1(z) + a$ at $z = 0$ satisfies the above conditions for all $a \in \mathbf{C}$ with sufficiently large modulus $|a|$.

Remark 3.6: Suppose that each a_i is a nonnegative real number and that there exists an S with $0 < S < R$ satisfying (2.3.1). Then the condition in Lemma 3.3 is equivalent to each of the following:

(1) $$\sum_{i=1}^{\infty}\frac{ia_i}{q^{i+1}} < 1;$$

(2) $$\sum_{i=1}^{\infty}(i-1)\frac{a_i}{q^i} < a_0;$$

(3) $$q^{-2}u'(q^{-1}) = q^{-2}h'(q^{-1}) < 1;$$

(4) $$e'(q^{-1}) < 2q,$$

where $e(x) = xh(x)$ for $x > R^{-1}$. Note that $h(q^{-1}) = q$ and $e(q^{-1}) = 1$. In particular, each of the above conditions is equivalent to (3.4.1).

Problem 3.7: Suppose that the sequence $\{a_i\}_{i=0}^{\infty}$ admits an S with $0 < S < R$ satisfying (2.3.1) or (2.3.2). If $\Sigma_{m=1}^{\infty}|b_m| \ge 1$, what happens? Does it happen that $\lim_{n\to\infty} g_n / q^n$ exists and is not equal to the value as in Proposition 3.4?

Remark 3.8: Suppose that each a_i is a nonnegative real number and that there exists an S with $0 < S < R$ satisfying (2.3.1). If $a = \lim_{n\to\infty} g_n / q^n$ exists, then we have $1 \le a\Sigma_{m=0}^{\infty} b_m \le 2$. This is seen as follows. First, we see easily that

$$\sum_{j=0}^{n}\sum_{m=0}^{j} b_m c_{j-m} \le \left(\sum_{m=0}^{n} b_m\right)\left(\sum_{l=0}^{n} c_l\right) \le \sum_{j=0}^{2n}\sum_{m=0}^{j} b_m c_{j-m}.$$

This implies that

$$n+1 \leq \left(\sum_{m=0}^{n} b_m\right)\left(\sum_{l=0}^{n} c_l\right) \leq 2n+1,$$

since $\sum_{m=0}^{j} b_m c_{j-m} = 1$ $(j \geq 0)$, as we have seen in the paragraph just before Lemma 3.2. Hence, we have

$$1 \leq \left(\sum_{m=0}^{n} b_m\right) \cdot \frac{1}{n+1}\left(\sum_{l=0}^{n} c_l\right) \leq \frac{2n+1}{n+1}.$$

Thus, if $a = \lim_{n\to\infty} c_n = \lim_{n\to\infty} g_n / q^n$ exists, then we have $1 \leq a\sum_{m=0}^{\infty} b_m \leq 2$.

Now we proceed to the study of the asymptotic behavior of the sequence $\{y_n\}_{n=1}^{\infty}$. By Lemma 3.1, for all $n \geq 1$, we have

$$y_n = g_n y_0 + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} g_{n-j} a_{i+j-1}\right) y_{-i}.$$

Thus, we have $d_n = c_n y_0 + \sum_{i=1}^{\infty} e_i^{(n)} y_{-i}$ $(n \geq 1)$, where $d_n = y_n / q^n$ and $e_i^{(n)} = \sum_{j=1}^{n}(c_{n-j} a_{i+j-1} / q^j)$. Since the above series converges absolutely by Lemma 3.1, we have

$$d_n = c_n y_0 + \sum_{j=1}^{n} \frac{c_{n-j}}{q^j}\left(\sum_{i=1}^{\infty} a_{i+j-1} y_{-i}\right) = c_n y_0 + \sum_{j=1}^{n} c_{n-j} p_j,$$

where $p_j = q^{-j}\sum_{i=1}^{\infty} a_{i+j-1} y_{-i}$ $(j \geq 1)$. Putting $p_0 = y_0$, we have $d_n = \sum_{j=0}^{n} c_{n-j} p_j$ $(n \geq 0)$. Set $t_n = d_n - d_{n-1}$ $(n \geq 1)$ and $t_0 = y_0 = d_0$. Then we have $t_n = p_0 k_n + p_1 k_{n-1} + \cdots + p_{n-1} k_1 + p_n k_0$ for all $n \geq 0$. Thus, if the series $\sum_{i=0}^{\infty} p_i$ converges absolutely, then the series $\sum_{n=0}^{\infty} t_n$ converges absolutely and is equal to the product $(\sum_{i=0}^{\infty} p_i)(\sum_{i=0}^{\infty} k_i)$, since the series $\sum_{i=0}^{\infty} k_i$ converges absolutely under the condition of Lemma 3.3. Note that $\sum_{i=0}^{n} t_i = d_n$ and that $\lim_{n\to\infty} d_n = \sum_{i=0}^{\infty} t_i$.

Lemma 3.9: If there exists an S with $0 < S < R$ satisfying (2.3.1) or (2.3.2), and (3.4.1), then the series $\sum_{i=0}^{\infty} p_i$ converges absolutely and is equal to $\sum_{i=0}^{\infty} q^i b_i y_{-i}$.

Proof: First, consider the series $\sum_{i=0}^{\infty} q^i b_i y_{-i}$. Since the sequence $\{y_{-i}\}_{i=0}^{\infty}$ is an element of X, there exist $C > 0$ and T with $0 < T < R$ satisfying $|y_{-i}| \leq CT^i$ for all i. If $T|q| \leq 1$, then we have

$$|q^i b_i y_{-i}| \leq C(T|q|)^i |b_i| \leq C|b_i|$$

and, hence, the series $\sum_{i=0}^{\infty} q^i b_i y_{-i}$ converges absolutely by the proof of Proposition 3.4. When $T|q| > 1$, we have

$$|q^i b_i y_{-i}| = |q|^i \left|\sum_{j=i}^{\infty} \frac{a_j}{q^{j+1}}\right| |y_{-i}| \leq |q|^i \left(\sum_{j=i}^{\infty} \frac{|a_j|}{|q|^{j+1}}\right) |y_{-i}| \leq C|q|^{-1}(T|q|)^i \sum_{j=i}^{\infty} \frac{|a_j|}{|q|^j}. \tag{3.9.1}$$

Now consider the series

$$\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(T|q|)^i\right)\frac{|a_j|}{|q|^j} = \sum_{j=0}^{\infty}\left(\frac{(T|q|)^{j+1}-1}{T|q|-1}\right)|a_j|(|q|^{-1})^j. \tag{3.9.2}$$

The radius of convergence of the power series

$$w(z)=\sum_{j=0}^{\infty}\left(\frac{(T|q|)^{j+1}-1}{T|q|-1}\right)|a_j|z^j$$

is equal to

$$\left(\limsup_{j\to\infty}\sqrt[j]{\left|\frac{(T|q|)^{j+1}-1}{T|q|-1}\right|}\sqrt[j]{|a_j|}\right)^{-1}=\frac{R}{T|q|}.$$

Since we always have $|q|^{-1}<R/(T|q|)$, we see that the series (3.9.2) converges (absolutely). Changing the order of the addition, we see that the series

$$\sum_{i=0}^{\infty}C|q|^{-1}(T|q|)^i\sum_{j=i}^{\infty}\frac{|a_j|}{|q|^j}$$

converges (absolutely). Thus, by (3.9.1), we see that the series $\sum_{i=0}^{\infty}q^ib_iy_{-i}$ converges absolutely.

Then we have

$$\sum_{i=1}^{\infty}q^ib_iy_{-i}=\sum_{i=1}^{\infty}q^i\left(\sum_{j=i}^{\infty}\frac{a_j}{q^{j+1}}\right)y_{-i}=\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{a_{i+j-1}}{q^j}\right)y_{-i}$$
$$=\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\frac{a_{i+j-1}}{q^j}y_{-i}\right)=\sum_{j=1}^{\infty}p_j.$$

Here we have changed the order of addition, which is allowed since the series in the first line converges absolutely as we have seen above. This completes the proof. □

Thus, we have proved the following.

Theorem 3.10: Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of complex numbers which satisfies (2.1) and which admits an S with $0<S<R$ satisfying (2.3.1) or (2.3.2), and (3.4.1). Then $\lim_{n\to\infty}y_n/q^n$ exists and is equal to

$$\left(\sum_{m=0}^{\infty}b_m\right)^{-1}\left(\sum_{i=0}^{\infty}p_i\right)=\frac{\sum_{m=0}^{\infty}b_mq^my_{-m}}{\sum_{m=0}^{\infty}b_m},$$

where q is as in Lemma 2.3.

Note that the above limiting value can be calculated by using only a_i $(i\geq 1)$, y_{-j} $(j\geq 0)$, and q. We also note that the above result coincides with the results in [2] concerning the case where there exists an integer r such that $a_i=0$ for all $i\geq r$. Furthermore, we note that, using our results of this section, we can obtain convergence results for the ratio of two ∞-generalized Fibonacci sequences and for the ratio of successive terms of an ∞-generalized Fibonacci sequence. For details, see [2, §3]. As to the ratio of two successive terms Dence [1] has obtained a similar result for weighted r-generalized Fibonacci sequences with r finite; however, Dence uses all the roots of the characteristic polynomial, while we obtain a formula in terms of only one root q.

Problem 3.11: For a given sequence $\{a_i\}_{i=0}^{\infty}$ as above, characterize those sequences $\{y_n\}_{n\in\mathbf{Z}}$ such that $y_n=f(y_{n-1},y_{n-2},y_{n-3},\dots)$ for all $n\in\mathbf{Z}$ (not just for $n\geq 1$). Note that $y_n=q^n$ is such an example. When $a_0=a_1=1$ and $a_i=0$ for all $i\geq 2$, then the sequence $\{y_n\}_{n\in\mathbf{Z}}$ defined by

$$y_n = \begin{cases} F_n & n \geq 1, \\ 0 & n = 0, \\ (-1)^{n+1} F_{-n} & n \leq -1, \end{cases}$$

is also such an example, where $\{F_n\}_{n=1}^{\infty}$ is the usual Fibonacci sequence.

4. EXAMPLES

In this section we give some examples that will help us to understand general phenomena.

Example 4.1: Let b and α be positive real numbers and consider the sequence $\{a_i\}_{i=0}^{\infty}$ defined by $a_i = b\alpha^i$. Then it is not difficult to see that $q = b + \alpha$, $\sum_{m=1}^{\infty} b_m = \alpha / b$, $g_0 = 1$, $g_1 = b$, and that $g_{n+1} = qg_n$ for all $n \geq 1$. Thus, we see that $\lim_{n\to\infty} g_n / q^n$ exists and is equal to $b / q = b / (b + \alpha) = (1 + \sum_{m=1}^{\infty} b_m)^{-1}$. This shows that, even if condition (3.4.1) is not satisfied, the conclusion of Proposition 3.4 holds in this case. In fact, condition (3.4.1) is equivalent to $\alpha / b < 1$ in this example. (When $\alpha / b < 1$, choose $r > 1$ with $r - 1 < b / \alpha < (r-1)^2$ and set $R = \alpha^{-1}$ and $S = (r\alpha)^{-1}$. Then condition (3.4.1) is satisfied.)

Example 4.2: We consider the sequence $\{a_i\}_{i=0}^{\infty}$ defined by $a_0 = 0$ and $a_i = b\alpha^i$ for $i \geq 1$ for some positive real numbers b and α, which is a slight modification of Example 4.1. It is easy to see that $q = (\alpha + \sqrt{\alpha^2 + 4b\alpha}) / 2$, $\sum_{m=1}^{\infty} b_m = b\alpha / (q - \alpha)^2 = b(b - (q - \alpha))^{-1} > 1$, $g_0 = 1$, $g_1 = 0$, and $g_{n+1} = \alpha g_n + b\alpha g_{n-1}$ for $n \geq 1$. Set $\xi_n = g_{n+1}$. Then we see that $\xi_{-1} = 1$, $\xi_0 = 0$, and $\xi_{n+1} = a_0'\xi_n + a_1'\xi_{n-1}$ $(n \geq 0)$, where $a_0' = \alpha$ and $a_1' = b\alpha$. Note that the number associated with the finite sequence $\{a_0', a_1'\}$ as in Lemma 2.3 coincides with the number q associated with $\{a_i\}_{i=0}^{\infty}$. Since conditions (2.1), (2.3.1), and (3.4.1) are satisfied for the sequence $\{a_0', a_1'\}$, we see that $\lim_{n\to\infty} \xi_n / q^n$ exists and is equal to $qb\alpha / (q^2 + b\alpha)$ by Theorem 3.10. Thus, we see that $\lim_{n\to\infty} g_n / q^n$ exists and is equal to $b\alpha / (q^2 + b\alpha)$. (This can also be obtained by a direct computation as in the previous example.) Note that we always have $\sum_{m=1}^{\infty} b_m = b(b - (q - \alpha))^{-1} > 1$ and that $(1 + \sum_{m=1}^{\infty} b_m)^{-1} = b\alpha / (q^2 + b\alpha)$. In other words, although condition (3.4.1) is not satisfied, the conclusion of Proposition 3.4 holds in this case.

Example 4.3: We consider the sequence $\{a_i\}_{i=0}^{\infty}$ defined by $a_i = a\alpha^i + b\beta^i$ for some positive real numbers a, b, α, and β. Then we see that $q > \alpha, \beta$ and that $q^2 - (\alpha + \beta + a + b)q + (b\alpha + a\beta + \alpha\beta) = 0$. Furthermore, we see that $g_0 = 1$, $g_1 = a + b$, $g_2 = (a+b)^2 + (a\alpha + g\beta)$, and $g_{n+1} = (\alpha + \beta + a + b)g_n - (b\alpha + a\beta + \alpha\beta)g_{n-1}$ for $n \geq 2$. Therefore, we have $g_n = Aq^n + Br^n$ $(n \geq 1)$ for some real numbers A and B, where r is the solution of the equation $r^2 - (\alpha + \beta + a + b)r + (b\alpha + a\beta + \alpha\beta) = 0$ with $r \neq q$. Since $|r| < q$, we see that $\lim_{n\to\infty} g_n / q^n$ exists and is equal to A. The value of A can be calculated by using g_1 and g_2. After tedious but elementary computations, we see that $A = (1 + a\alpha / (q - \alpha)^2 + b\beta / (1 - \beta)^2)^{-1} = (1 + \sum_{m=1}^{\infty} b_m)^{-1}$. Note that the value $\sum_{m=1}^{\infty} b_m$ can be greater than 1. For example, for $(\alpha, \beta, a, b) = (1, 1/2, 1, 1)$, the sum is smaller than 1 while, for $(\alpha, \beta, a, b) = (3, 1, 1, 1)$, it is greater than 1.

Example 4.4: Consider the sequence $\{a_i\}_{i=0}^{\infty}$ with $a_i = 1 / (i+1)!$. Note that, for this sequence, we have $h(x) = (e^x - 1) / x$ and $e(x) = e^x - 1$. Hence, the radius of convergence R is equal to ∞. In this case, we can easily check that $q = (\log 2)^{-1}$. Hence, we have $e'(q^{-1}) = 2 < 2(\log 2)^{-1} = 2q$, which implies that the condition in Lemma 3.3 is satisfied by Remark 3.6. Thus, by an easy calculation, we see that the sequence $\{g_n\}_{n=1}^{\infty}$ behaves like $(\log 2)^{-(n+1)}/2$ when n goes to ∞. More generally, the sequence $\{y_n\}_{n=1}^{\infty}$ behaves like $b(\log 2)^{-n}$, where

$$b = \frac{1}{2}(\log 2)^{-1}\left(y_0 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(\log 2)^j}{(i+j)!}y_{-i}\right).$$

Problem 4.5: For an ∞-generalized Fibonacci sequence, the function $H(z) = z^{-1}h(z^{-1}) - 1$ seems to be the analog to the characteristic polynomial in the finite case. This raises the question as to a possible analog to Binet-type formulas for the finite case (see [3] and [2, Th. 1], for example). If $H(z)$ has finitely many zeros, Examples 4.1 through 4.3 seem to suggest that Binet-type formulas hold as in the finite case. If $H(z)$ has infinitely many zeros, as in Example 4.4, then will there be a Binet-type formula that is an infinite series involving powers of the zeros?

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