GENERALIZED BRACKET FUNCTION INVERSE PAIRS

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The aim of this paper is to prove the existence of inverse pairs for a certain class of number-theoretic functions. An application of the result is also illustrated. The motivation comes from the study of functions such as

$$C_k(n) = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i\geq 1}} 1 \quad \text{and} \quad R_k(n) = \sum_{\substack{a_1+\cdots+a_k=n \\ (a_1,\ldots,a_k)=1}} 1 .$$

Gould [1] showed that $C_k(n) = \Sigma_{d|n} R_k(d)$ and that $R_k(n)$ has an inverse. In [5] a pair of functions similarly related is also studied and similar results obtained.

We start our investigation by giving the following theorem due to Gould [2].

Theorem 1 (The Bracket Function Transform): Define

$$S(n) = \sum_{k=1}^{n} \left[\frac{n}{k}\right] A_k = \sum_{j=1}^{n} \sum_{d|j} A_d, \tag{1}$$

$$A(x) = \sum_{n=1}^{\infty} x^n A_n, \tag{2}$$

and

$$S(x) = \sum_{n=1}^{\infty} x^n S_n. \tag{3}$$

Then

$$S(x) = \frac{1}{1-x} \sum_{n=1}^{\infty} A_n \frac{x^n}{1-x^n}. \tag{4}$$

From this it follows that

$$(1-x)\,S(x) = \sum_{n=1}^{\infty} x^n S_n - \sum_{n=1}^{\infty} x^{n+1} S_n = \sum_{n=1}^{\infty} (S_n - S_{n-1})\, x^n, \text{ where } S_0 = 0. \tag{5}$$

That is

$$\sum_{n=1}^{\infty} (S_n - S_{n-1})\, x^n = \sum_{n=1}^{\infty} A_n \frac{x^n}{(1-x^n)},$$

a result equivalent to

$$S_n - S_{n-1} = \sum_{d|n} A_d \quad \text{(see Hardy \& Wright [4], p. 257).} \tag{6}$$

But relation (6), in turn, implies that

$$A(n) = \sum_{d|n} (S(d) - S(d-1))\,\mu\!\left(\frac{n}{d}\right). \tag{7}$$

A result also obtained by Gould [2], albeit through an entirely different argument. For completeness, we also include here Gould's [2] elegant formulation of the above result.

Theorem 2:

$$a(n,k)=\sum_{j=1}^{n}\left[\frac{n}{j}\right]b(j,k)=\sum_{j=1}^{n}\sum_{d\mid j}b(d,k)$$

if and only if

$$b(n,k)=\sum_{d\mid n}(a(d,k)-a(d-1,k))\mu\left(\frac{n}{d}\right).$$

We now prove our next result.

Lemma 3: Define

$$H(x)=\sum_{n=1}^{\infty}x^nH_n,\quad \text{where } H_n=\sum_{d\mid n}A_d.$$

Then

$$S(x)=\frac{H(x)}{1-x}. \tag{8}$$

Proof:

$$S(x)=\sum_{n=1}^{\infty}S_nx^n=\sum_{n=1}^{\infty}x^n\sum_{j=1}^{n}\sum_{d\mid j}A_d=\sum_{n=1}^{\infty}x^n\sum_{j=1}^{n}H_j$$
$$=\sum_{j=1}^{\infty}H_j\sum_{n=j}^{\infty}x^n=\sum_{j=1}^{\infty}H_jx^j\sum_{n=j}^{\infty}x^{n-j}=\frac{H(x)}{1-x},\ |x|<1.$$

Next, we prove our main result.

Theorem 4: Define

$$H(n,k)=\sum_{d\mid n}A(d,k),$$
$$S(n,k)=\sum_{j=k}^{n}\left[\frac{n}{j}\right]A(j,k)=\sum_{j=k}^{n}\sum_{d\mid j}A(d,k),\ \text{and}$$
$$B(n,k)=\left[\frac{n}{k}\right]-\sum_{j=k}^{n-1}S(n,j)B(j,k),$$

where $A(n,k)=B(n,k)=0$ if $n<k$ and $A(k,k)=B(k,k)=1$.

Then the functions $A(n,k)$ and $B(n,k)$ satisfy the orthogonality relations

$$\sum_{j=k}^{n}A(j,k)B(n,j)=\delta_k^n \text{ and } \sum_{j=k}^{n}B(j,k)A(n,j)=\delta_k^n,\ \text{where } \delta_k^n=\begin{cases}1, & \text{if } n=k,\\ 0, & \text{otherwise.}\end{cases}$$

Proof: Consider

$$\sum_{n=1}^{\infty}B(n,k)x^n=\sum_{n=1}^{\infty}\left[\frac{n}{k}\right]x^n-\sum_{n=1}^{\infty}x^n\sum_{j=k}^{n-1}S(n,j)B(j,k)$$

$$= \frac{x^k}{(1-x)(1-x^k)} + \sum_{j=k}^{\infty} B(j, k)x^j - \sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} S(n, j)B(j, k),$$

since $S(k, k) = 1$ by hypothesis.

That is,

$$\sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} S(n, j)B(j, k) = \frac{x^k}{(1-x)(1-x^k)} \tag{9}$$

or

$$\sum_{j=k}^{\infty} B(j, k) \sum_{n=j}^{\infty} x^n S(n, j) = \sum_{j=k}^{\infty} B(j, k)S(x) = \frac{x^k}{(1-x)(1-x^k)}. \tag{10}$$

From the last equality in (10) and Theorem 1, we have

$$\sum_{j=k}^{\infty} B(j, k) \sum_{n=j}^{\infty} A(n, j) \frac{x^n}{1-x^n} = \sum_{n=k}^{\infty} \sum_{j=k}^{n} B(j, k)A(n, j) \frac{x^n}{1-x^n} = \frac{x^k}{1-x^k},$$

from which it follows that

$$\sum_{j=k}^{n} B(j, k)A(n, j) = \delta_k^n.$$

Also, from $H(n, k) = \sum_{d|n} A(d, k)$, Theorem 1, and Lemma 3, we have that

$$\sum_{n=1}^{\infty} A(n, k) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} H(n, k)x^n,$$

and, hence, that

$$\sum_{n=1}^{\infty} A(n, k) \frac{x^n}{(1-x)(1-x^n)} = \frac{H(x)}{1-x}.$$

We may now use relation (9) and rewrite this last equation in the form

$$\sum_{n=1}^{\infty} A(n, k) \sum_{i=n}^{\infty} x^i \sum_{j=n}^{i} S(i, j)B(j, n) = \sum_{n=1}^{\infty} S(n, k)x^n,$$

that is,

$$\sum_{j=1}^{\infty} \sum_{n=1}^{j} B(j, n)A(n, k) \sum_{i=j}^{\infty} x^i S(i, j) = \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} x^i S(i, j) \sum_{n=1}^{j} B(j, n)A(n, k)$$
$$= \sum_{n=1}^{\infty} S(n, k)x^n,$$

which implies that $\sum_{n=1}^{j} B(j, n)A(n, k) = \delta_k^j$, and, hence, the result $\sum_{j=1}^{n} A(j, k)B(n, j) = \delta_k^n$..

Theorem 4, in turn, implies the following result.

Theorem 5: For any ordered pair of functions $\langle f(n, k), g(n, k) \rangle$, the following holds:

$$f(n, k) = \sum_{j=k}^{n} g(n, j)A(j, k) \text{ if and only if } g(n, k) = \sum_{j=k}^{n} f(n, j)B(j, k),$$

where $A(n, k)$ and $B(n, k)$ are as defined in Theorem 4.

Of interest are the function pairs $\langle f(n, k), g(n, k)\rangle$ satisfying Theorem 5. One such class may be obtained from the following result.

Theorem 6: Let

$$g(n, k) = \begin{cases} h(n, k), & \text{if } k / n, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(n, k) = \sum_{d|n} h(n, d) A(d, k).$$

Then $\langle f(n, k), g(n, k)\rangle$ satisfies Theorem 5, where $h(n, k)$ is any number-theoretic function.

Proof: If $f(n, k) = \sum_{d|n} h(n, d) A(d, k)$, then

$$\begin{aligned} g(n, k) &= \sum_{j=k}^{n} \sum_{d|n} h(n, d) A(d, j) B(j, k) = \sum_{j=k}^{n} \sum_{\substack{d=j \\ d|n}}^{n} h(n, d) A(d, j) B(j, k) \\ &= \sum_{\substack{d=k \\ d|n}}^{n} \sum_{j=k}^{d} h(n, d) A(d, j;) B(j, k) = \sum_{\substack{d=k \\ d|n}}^{n} h(n, d) \delta_k^d \\ &= \begin{cases} h(n, k), & \text{if } k / n, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The converse is trivial.

Similarly, it may be shown that the functions

$$f(n, k) = \begin{cases} h(n, k), & \text{if } k / n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(n, k) = \sum_{d|n} h(n, d) B(d, k)$$

satisfy Theorem 5.

As an application, we shall consider some of the results in [5]. There it was established that

$$\binom{n}{k} = \sum_{d|n} T_k^d(d), \tag{11}$$

where

$$T_k^n(n) = \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_k \le n \\ (a_1, a_2, \ldots, a_k, n) = 1}} 1 \quad , \quad n \ge k. \tag{12}$$

It follows from equation (11) that

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1} = \sum_{j=k}^{n} \sum_{d|j} T_k^d(d) = \sum_{j=k}^{n} \left[\frac{n}{j}\right] T_k^j(j). \tag{13}$$

Therefore, by Theorem 2,

$$T_k^n(n) = \sum_{d|n} \left\{ \binom{d+1}{k+1} - \binom{d}{k+1} \right\} \mu\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{n}{k}. \tag{14}$$

Further, if $A(n,k)=T_k^n(n)$ and $H(n,k)=\sum_{d|n}T_k^d(d)=\binom{n}{k}$, we may also apply Theorem 1 and Lemma 3 to find the equivalent $S(x)$ and $H(x)$.

And, by Theorem 4, the function $T_k^n(n)$ has an inverse given by

$$K_k(n)=\left[\frac{n}{k}\right]-\sum_{j=k}^{n-1}\binom{n+1}{j+1}K_k(j). \tag{15}$$

Clearly,

$$K_k(k)=\left[\frac{k}{k}\right],\quad K_k(k+1)=\binom{k+2}{k+2}\left[\frac{k+1}{k}\right]-\binom{k+2}{k+1}\left[\frac{k}{k}\right],$$

and

$$\begin{aligned}K_k(k+2)&=\binom{k+3}{k+3}\left[\frac{k+2}{k}\right]-\binom{k+3}{k+2}\left[\frac{k+1}{k}\right]+\left[\frac{k}{k}\right]\left\{\binom{k+3}{k+2}\binom{k+2}{k+1}-\binom{k+3}{k+1}\right\}\\&=\binom{k+3}{k+3}\left[\frac{k+2}{k}\right]-\binom{k+3}{k+2}\left[\frac{k+1}{k}\right]+\binom{k+3}{k+1}\left[\frac{k}{k}\right]\\&=\sum_{j=k}^{k+2}\binom{(k+2)+1}{j+1}\left[\frac{j}{k}\right](-1)^{(k+2)-j}.\end{aligned}$$

We may, therefore, generalize and obtain the following explicit form for $K_k(n)$.

Theorem 7:

$$K_k(n)=\sum_{j=k}^{n}(-1)^{n-j}\binom{n+1}{j+1}\left[\frac{j}{k}\right]\quad\text{where } K_k(n)=0 \text{ if } n<k.$$

Proof: We prove the result by induction on n. We shall assume the result holds for k, $k+1$, ..., n and consider

$$\begin{aligned}K_k(n+1)&=\left[\frac{n+1}{k}\right]-\sum_{j=k}^{n}\binom{n+2}{j+1}K_k(j)\\&=\left[\frac{n+1}{k}\right]-\sum_{j=k}^{n}\binom{n+2}{j+1}\sum_{i=k}^{j}(-1)^{j-i}\binom{j+1}{i+1}\left[\frac{i}{k}\right]\quad\text{by the inductive hypothesis}\\&=\left[\frac{n+1}{k}\right]-\sum_{i=k}^{n}\sum_{j=i}^{n}(-1)^{j-i}\binom{n+2}{j+1}\binom{j+1}{i+1}\left[\frac{i}{k}\right]\\&=\left[\frac{n+1}{k}\right]-\sum_{i=k}^{n}\left[\frac{i}{k}\right]\sum_{j=i+1}^{n+2}\binom{n+2}{j}\binom{j}{i+1}(-1)^{j-(i+1)}-\sum_{i=k}^{n}\left[\frac{i}{k}\right]\binom{n+2}{i+1}(-1)^{n-i}\\&=\sum_{j=k}^{n+1}(-1)^{n+1-j}\binom{n+2}{j+1}\left[\frac{j}{k}\right],\end{aligned}$$

where we have used the identity

$$\sum_{j=i}^{n}(-1)^{j-i}\binom{n}{j}\binom{j}{i}=\delta_i^n\quad\text{(see Gould [3, (3.119)], p. 36).}$$

And, from Theorem 5, it follows that

$$f(n,k)=\sum_{j=k}^{n} g(n,j)T_k^j(j) \text{ if and only if } g(n,k)=\sum_{j=k}^{n} f(n,j)K_k(j). \tag{16}$$

The functions

$$f(n,k)=\binom{n+1}{k+1} \quad\text{and}\quad g(n,k)=\left[\frac{n}{k}\right]$$

are particular cases of this result.

Also, from Theorem 6, we may obtain other such function pairs for given $h(n,k)$; in particular, with $h(n,k)=1$, we obtain the functions

$$g(n,k)=\begin{cases}1, & \text{if } k/n,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$f(n,k)=\sum_{d\mid n} T_k^d(d)=\binom{n}{k}.$$

Further, using the techniques in [6], we may prove the following result.

Theorem 8: Let

$$f(n,k)=\frac{(-1)^k}{k+1}$$

and

$$g(n,k)=\sum_{i=k}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{k}\right]\binom{n+1}{i+1}.$$

Then $\langle f(n,k), g(n,k)\rangle$ satisfies Theorem 5, where $A(n,k)=T_k^n(n)$ and $B(n,k)=K_k(n)$.

Proof: Suppose $f(n,k)=\frac{(-1)^k}{k+1}$, then

$$g(n,k)=\sum_{j=k}^{n} f(n,j)\sum_{i=k}^{j}(-1)^{j-i}\binom{j+1}{i+1}\left[\frac{i}{k}\right]=\sum_{i=k}^{n}(-1)^i\left[\frac{i}{k}\right]\sum_{j=i}^{m}\frac{(-1)^{2j}}{j+1}\binom{j+1}{i+1}$$
$$=\sum_{i=k}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{k}\right]\sum_{j=i}^{n}\binom{j}{i}=\sum_{i+1}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{k}\right]\binom{n+1}{i+1}.$$

Conversely, assuming this form for $g(n,k)$, we obtain that

$$f(n,k)=\sum_{j=k}^{n}\sum_{i=j}^{n}\frac{(-1)^i}{i+1}\left[\frac{i}{j}\right]\binom{n+1}{i+1}T_k^j(j)=\sum_{i=k}^{n}\frac{(-1)^i}{i+1}\binom{n+1}{i+1}\sum_{j=k}^{i}\left[\frac{i}{j}\right]T_k^j(j)$$
$$=\sum_{i=k}^{n}\frac{(-1)^i}{i+1}\binom{n+1}{i+1}\binom{i+1}{k+1}=\frac{1}{k+1}\sum_{i=k}^{n}(-1)^i\binom{n+1}{i+1}\binom{i}{k}$$

from equation (13). We now use the relation

$$\sum_{j=i}^{n}\binom{j}{i}=\binom{n+1}{i+1} \quad \text{(see Gould [3, (1.52)])}$$

to obtain that

$$\sum_{i=k}^{n}(-1)^i\binom{n+1}{i+1}\binom{i}{k}=\sum_{i=k}^{n}(-1)^i\binom{i}{k}\sum_{j=i}^{n}\binom{j}{i}$$
$$=\sum_{j=k}^{n}(-1)^k\sum_{i=k}^{j}(-1)^{k-i}\binom{j}{i}\binom{i}{k}=(-1)^k$$

and, hence, the result.

Clearly, many more such function pairs can be found by use of the right Binomial Identities. And, as in [6], generalizations of such functions are also possible.

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