PARTIAL FIBONACCI AND LUCAS NUMBERS

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0. INTRODUCTION

The well-known Lucas formula

$$F_{n+1} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} \tag{1}$$

connects the Fibonacci numbers with binomial coefficients. Our interest is to find out what kind of numbers are obtained by taking every number r in (1) from a fixed residue class modulo m $(m = 2, 3, \ldots)$. As a result, a new family of sequences is introduced: the partial, or $1/m$-Fibonacci numbers. We give here a primary description of these numbers and their generating functions. By a similar construction, partial Lucas, Pell, and other specialized Fibonacci-type sequences can be obtained. Properties of these number systems will be explained in many respects.

1. THE BASIC RECURSION

Given a modulo m $(m = 1, 2, 3, \ldots)$, we define the (m, k)*-Fibonacci numbers* as follows:

$$F_{n+1}^{(m,k)} = \sum_{r=0}^{\ell} \binom{n-mr-k}{mr+k} \quad (k = 0, 1, \ldots, m-1), \tag{2}$$

where $\ell = \lfloor (n-2k)/2m \rfloor$; $n = 2k, 2k+1, \ldots$. For $n = 1, \ldots, 2k$, $F_n^{(m,k)} = 0$ $(k > 0)$. Irrespective of the value of k or even of m, these numbers may be called *1/m-Fibonacci numbers* or *partial Fibonacci numbers*. For every natural n, according to (1),

$$\sum_{k=0}^{m-1} F_n^{(m,k)} = F_n = F_n^{(1,0)}. \tag{3}$$

For $n \le 2m$, there is $F_n^{(m,k)} = \binom{n-k-1}{k}$ for all k. We usually disregard (except in §4) the all-zero case $n = 0$.

Theorem 1: For every m, the sequence $\{F_n^{(m,k)}\}$ is the difference sequence of $\{F_n^{(m,k+1)}\}$ over k in cyclic order, i.e.,

$$\begin{aligned} F_{n+1}^{(m,k)} &= F_{n+3}^{(m,k+1)} - F_{n+2}^{(m,k+1)} \quad (k < m-1), \\ F_{n+1}^{(m,m-1)} &= F_{n+3}^{(m,0)} - F_{n+2}^{(m,0)} \quad (k = m-1). \end{aligned} \tag{4}$$

Proof: As $\binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k}$, for the r^{th} summand in (2) there obviously is

$$\binom{n-mr-k}{mr+k} = \binom{n+2-mr-k-1}{mr+k+1} - \binom{n+1-mr-k-1}{mr+k+1} \quad (k < m-1),$$

$$\binom{n-mr-m+1}{mr+m-1} = \binom{n+2-m(r+1)}{m(r+1)} - \binom{n+1-m(r+1)}{m(r+1)} \quad (k = m-1).$$

In the last case, for $r = 0$ the right side is

$$\binom{n+2}{0} - \binom{n+1}{0} = 0. \quad \square$$

Thus, all m sequences $\{F_n^{(m,k)}\}$ form a cyclic set with respect to the difference operator Δ_2 (see [3]).

Theorem 2: For every m and k, the recurrence

$$F_n^{(m,k)} = \sum_{s=0}^{m} (-1)^s \binom{m}{s} F_{n+2m-s}^{(m,k)} \tag{5}$$

of order $2m$ holds.

Proof: From (4), with n instead of $n+1$, by consecutive forward substitutions

$$F_n^{(m,k+1)} \to F_n^{(m,k)} \ (k < m-1), \quad F_n^{(m,0)} \to F_n^{(m,m-1)},$$

and with $k = 0$ instead of $k = m$ for the transition step (addition modulo m), we have

$$\begin{aligned} F_n^{(m,k)} &= F_{n+4}^{(m,k+2)} - 2F_{n+3}^{(m,k+2)} + F_{n+2}^{(m,k+2)} \\ &= F_{n+6}^{(m,k+3)} - 3F_{n+5}^{(m,k+3)} + 3F_{n+4}^{(m,k+3)} - F_{n+3}^{(m,k+3)} = \cdots, \end{aligned}$$

so that (5) follows after $m-1$ steps. This can be proved easily by induction. $\square$

2. FIBONACCI CYCLOTOMIC POLYNOMIALS

From the recurrence (5), we obtain the characteristic polynomial

$$\sum_{s=0}^{m} (-1)^s \binom{m}{s} x^{2m-s} - 1 = (x^2 - x)^m - 1 = p_m(x) \tag{6}$$

of degree $2m$. The polynomials (6) can be called *Fibonacci cyclotomic polynomials*, as the substitution $u = x(x-1)$ turns them into the classical cyclotomic polynomials (see [4]). Hence, they admit the following factorization over $\mathbb{C}$:

$$p_m(x) = \prod_{j=0}^{m-1} (x^2 - x - \varepsilon^j), \tag{7}$$

where $\varepsilon^j = \cos\frac{2\pi j}{m} + i \sin\frac{2\pi j}{m}$ are the values of $\sqrt[m]{1}$. The factor $x^2 - x - 1$ (for $j = 0$) whose zeros are $\alpha = \frac{1}{2}(1+\sqrt{5})$, $\beta = 1 - \alpha$, is present in all $p_m(x)$. The quotient polynomial

$$q_m(x) = \frac{p_m(x)}{x^2 - x - 1} = \sum_{j=0}^{m-1} (x^2 - x)^j \tag{8}$$

has the first m lower terms $(-1)^h F_{h+1} x^h$ $(h = 0, 1, \ldots, m-1)$ and its (pairwise conjugate) zeros are

$$\begin{aligned} &\zeta_j,\ \bar{\zeta}_j = \tfrac{1}{2}\left(1 \pm \sqrt{1 + 4\varepsilon_j}\right); \\ &|\zeta_j| = \sqrt{17 + 8\cos\frac{2\pi j}{m}} \quad (j = 1, 2, \ldots, m-1). \end{aligned} \tag{9}$$

Examples:

$$q_1(x)=1;\quad q_2(x)=x^2-x+1;\quad q_3(x)=x^4-2x^3+2x^2-x+1;$$
$$q_4(x)=x^6-3x^5+4x^4-3x^3+2x^2-x+1=q_2(x)(x^4-2x^3+x^2+1);$$
$$q_5(x)=x^8-4x^7+7x^6-7x^5+5x^4-3x^3+2x^2-x+1;$$
$$q_6(x)=x^{10}-5x^9+11x^8-14x^7+12x^6-8x^5+5x^4-3x^3+2x^2-x+1$$
$$=q_2(x)q_3(x)(x^4-2x^3+x+1).$$

The final factorization to quadratic trinomials over $\mathbb{R}$ is more difficult:

$$q_3(x)=(x^2-(1+A)x+M)(x^2-(1-A)x+1/M),$$
$$\frac{q_4(x)}{q_2(x)}=(x^2-(1+B)x+N)(x^2-(1-B)x+1/N),$$

where

$$A=\sqrt{\tfrac{1}{2}(\sqrt{13}-1)},\qquad M=\tfrac{1}{4}\left(\sqrt{13}+1+\sqrt{2(\sqrt{13}-1)}\right);$$
$$B=\sqrt{\tfrac{1}{2}(\sqrt{17}+1)},\qquad N=\tfrac{1}{4}\left(\sqrt{17}+1+\sqrt{2(\sqrt{17}+1)}\right).$$

Solutions of the equation $q_m(x)=0$ for $m\le 6$ involve radicals $\sqrt{3}$, $\sqrt{5}$, $\sqrt{13}$, $\sqrt{17}$, and $\sqrt{21}$.

3. GENERATING FUNCTIONS

Theorem 3: The generating function of the sequence $\{F_n^{(m,k)}\}$,

$$f^{(m,k)}(x)=\sum_{n=2k}^{\infty}F_{n+1}^{(m,k)}x^n=\frac{x^{2k}(1-x)^{m-k-1}}{r_m(x)},\tag{10}$$

where

$$r_m(x)=x^{2m}p_m(1/x)=(1-x)^m-x^{2m}.$$

Proof: In the case $k=m-1$,

$$f^{(m,m-1)}(x)=\frac{x^{2m-2}}{r_m(x)},\tag{11}$$

i.e., the series $\sum_{n=0}^{\infty}F_{2m+n+1}^{(m,m-1)}x^n$ with shifted coefficient sequence (with $F_{2m+1}^{(m,m-1)}=1$ being the first one) is the inverse for $r_m(x)$:

$$\frac{1}{x^{2m-2}}f^{(m,m-1)}(x)r_m(x)=1,$$

as can be seen from the convolution formulas (see [2], [3])

$$\sum_{j=0}^{\ell}(-1)^j\binom{m}{j}\binom{m+\ell-j-1}{m-1}=\begin{cases}1 & (\ell=0),\\ 0 & (\ell=1,\dots,m).\end{cases}$$

Further, it follows from (4) that

$$f^{(m,k)}(x) = \frac{1-x}{x^2} f^{(m,k+1)}(x) \quad (k = 0, 1, \dots, m-2). \tag{12}$$

From this, we obtain (10). In particular,

$$f^{(m,0)}(x) = \frac{(1-x)^{m-1}}{r_m(x)}. \quad \square \tag{13}$$

Now we can verify the identity (3) in terms of generating functions. Indeed,

$$r_m(x) = (1-x-x^2)s_m(x),$$

where

$$s_m(x) = x^{2m} q_m(1/x) = \sum_{k=0}^{m-1} x^{2k}(1-x)^{m-k-1}$$

is exactly the sum of numerators in (10) over all k. Hence,

$$\sum_{k=0}^{m-1} f^{(m,k)}(x) = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1}x^n = f(x).$$

4. EXPLICIT EXPRESSIONS: $m = 2$

In some simplest cases, it is possible to express the numbers $F_n^{(m,k)}$ directly as functions of n, thus giving generalizations of the Binet formula

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \tag{14}$$

For $m = 2$, denote

$$F_n^{(2,0)} = \sum_{r=0}^{\lfloor (n-1)/4 \rfloor} \binom{n-1-2r}{2r} = E_n$$

and

$$F_n^{(2,1)} = \sum_{r=0}^{\lfloor (n-3)/4 \rfloor} \binom{n-2-2r}{2r+1} = D_n$$

(the *even* and *odd semi-Fibonacci numbers*). Then, from (6) and (7), the characteristic equation

$$p_2(x) \equiv (x^2-x-1)(x^2-x+1) = 0$$

is obtained, whose roots are $\alpha, \beta = 1-\alpha$, and $\varepsilon, \bar{\varepsilon} = \frac{1}{2}(1 \pm i\sqrt{3})$. As $\varepsilon^6 = 1$, there is

$$\varepsilon^2 = \varepsilon - 1, \quad \varepsilon^3 = -1, \quad \varepsilon^4 = -\varepsilon, \quad \varepsilon^5 = 1 - \varepsilon = \bar{\varepsilon}.$$

Using the (extended) initial conditions $E_0 = D_0 = D_1 = D_2 = 0$ and $E_1 = E_2 = E_3 = D_3 = 1$ in the general solution

$$E_n, D_n = A\alpha^n + B(1-\alpha)^n + C\varepsilon^n + D(1-\varepsilon)^n,$$

we obtain for both E_n and D_n,

$$A = -B = \frac{2\alpha - 1}{10} = \frac{1}{2(2\alpha-1)} = \frac{1}{2\sqrt{5}},$$

and for E_n and D_n, respectively (instead of C and D),

$$C' = -D' = \frac{1}{2(2\varepsilon-1)} \quad \text{and} \quad C'' = -D'' = -\frac{1}{2(2\varepsilon-1)}.$$

Hence,

$$E_n, D_n = \frac{1}{2(2\alpha-1)}(\alpha^n - (1-\alpha)^n) \pm \frac{1}{2(2\varepsilon-1)}(\varepsilon^n - (1-\varepsilon)^n), \tag{15}$$

and, in accordance to (3), $E_n + D_n = F_n$. The first summand in (15) is exactly $F_n/2$, whereas the differences

$$\delta_n = E_n - D_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-r-1}{r} = \frac{1}{2\varepsilon-1}(\varepsilon^n - (1-\varepsilon)^n)$$

form a periodic sequence $(0, 1, 1, 0, -1, -1)$ modulo 6. (See also [1].)

The generating functions (11) and (13) are

$$f^{(2,0)}(x) = \sum_{n=0}^{\infty} E_{n+1} x^n = (1-x)/r_2(x) = e(x)$$

and

$$f^{(2,1)}(x) = \sum_{n=0}^{\infty} D_{n+1} x^n = x^2/r_2(x) = d(x),$$

where $r_2(x) = (1-x-x^2)(1-x+x^2)$. Then

$$e(x) + d(x) = \frac{1}{1-x-x^2} = f(x),$$

$$e(x) - d(x) = \frac{1}{1-x+x^2} = \sum_{n=0}^{\infty} (x-x^2)^n = 1 + x - x^3 - x^4 + x^6 + x^7 - \cdots.$$

5. PARTIAL LUCAS NUMBERS

Next we apply our approach to the Lucas numbers

$$L_n = F_{n-1} + F_{n+1} = 1 + \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \left(\binom{n-r-1}{r-1} + \binom{n-r}{r} \right). \tag{16}$$

Then a definition of the (m, k)-*Lucas numbers*, parallel to (2), is

$$L_{n+1}^{(m,k)} = 1 + \sum_{r=0}^{\ell} \left(\binom{n-mr-k}{mr+k} + \binom{n-mr-k+1}{mr+k+1} \right) \quad (k = 0, 1, \ldots, m-1), \tag{17}$$

where $\ell = \lfloor (n-2k)/2m \rfloor$; $n = 2k, 2k+1, \ldots$. For $n = 0, 1, \ldots, 2k$, $L_n^{(m,k)} = 0$ $(k > 0)$, and $L_0^{(m,0)} = 2$, $L_0^{(m,k)} = 0$ $(k > 0)$. The formula

$$\sum_{k=0}^{m-1} L_n^{(m,k)} = L_n = L_n^{(1,0)} \tag{18}$$

corresponds to (3).

The numbers $L_n^{(m,k)}$ satisfy conditions analogous to (4) and, consequently, also the basic recursion (5). The particular solutions differ from the previous Fibonacci case only because of another initial conditions. Thus, for $m=2$ (the *semi-Lucas numbers*), we obtain, instead of (15),

$$L_n^{(2,0)},\ L_n^{(2,1)} = \tfrac{1}{2}L_n \pm \tfrac{1}{2}(\varepsilon^n + (1-\varepsilon)^n). \tag{19}$$

The differences $\delta'_n = L_n^{(2,0)} - L_n^{(2,1)}$ form a periodic sequence $(2, 1, -1, -2, -1, 1)$ modulo 6. The generating functions are

$$\ell^{(2,0)}(x) = \sum_{n=0}^{\infty} L_{n+1}^{(2,0)} x^n = \frac{2-3x+x^2}{r_2(x)}$$

and

$$\ell^{(2,1)}(x) = \sum_{n=0}^{\infty} L_{n+1}^{(2,1)} x^n = \frac{2x^2-x^3}{r_2(x)},$$

and their sum (18) is

$$\frac{2-x}{1-x-x^2} = \sum_{n=0}^{\infty} L_{n+1} x^n = \ell(x).$$

The general formula that corresponds to (10) here is

$$\ell^{(m,k)}(x) = \sum_{n=2k}^{\infty} L_{n+1}^{(m,k)} x^n = \frac{x^{2k}(1-x)^{m-k-1}(2-x)}{r_m(x)}. \tag{20}$$

6. NUMERICAL RESULTS

We give the values of $F_n^{(m,k)}$ and $L_n^{(m,k)}$ for $m \le 4$ in Tables 1 and 2 below. For the negative subscripts (in Table 1), formulas (4) were used.

7. SOME PROPERTIES

We mention here without proof the following appealing properties of $F_n^{(m,k)}$ and $L_n^{(m,k)}$, discovered after short observations:

1) $F_{-n}^{(m,k)} = (-1)^{n+1} F_n^{(m,k\ominus r)}$; (21)

2) $L_{-n}^{(m,k)} = (-1)^n L_n^{(m,k\ominus r)}$ $(n = mq + r > 0,\ r = 0, 1, \ldots, m-1)$, (22)
where $\ominus$ is subtraction modulo m;

3) $L_n^{(m,k)} = F_{n-1}^{(m,k\ominus 1)} + F_{n+1}^{(m,k)}$; (23)

4) $L_n^{(m,k)} = F_{n+2}^{(m,k)} - F_{n-2}^{(m,k\oplus(m-2))}$, (24)
where $\oplus$ is addition modulo m;

5) $$\sum_{j=1}^{n} F_j^{(m,k)} = \begin{cases} F_{n+2}^{(m,k+1)} & (k = 0, 1, \ldots, m-2), \\ F_{n+2}^{(m,0)} - 1 & (k = m-1); \end{cases} \tag{25}$$

6) $$\sum_{j=1}^{n} L_j^{(m,k)} = \begin{cases} L_{n+2}^{(m,1)} - 2 & (k = 0); \\ L_{n+2}^{(m,k+1)} & (k = 1, \ldots, m-2), \\ L_{n+2}^{(m,0)} - 1 & (k = m-1). \end{cases} \tag{26}$$

These examples reveal a remarkable variety of repetition patterns, including the "rotation" (twisting) phenomenon. The usual Fibonacci-type formulas are obtained by summation over all k.

TABLE 1. Numbers $F_n^{(m,k)}$

n	F_n	$m=2$		δ_n	$m=3$			$m=4$			
		$k=0$	1		$k=0$	1	2	$k=0$	1	2	3
−10	−55	−27	−28	1	−13	−21	−21	−21	−20	−6	−8
−9	34	17	17	0	11	8	15	7	15	10	2
−8	−21	−11	−10	−1	−10	−5	−6	−1	−6	−10	−4
−7	13	6	7	−1	5	6	2	1	1	5	6
−6	−8	−4	−4	0	−1	−4	−3	−3	0	−1	−4
−5	5	3	2	1	1	1	3	3	1	0	1
−4	−3	−1	−2	1	−2	0	−1	−1	−2	0	0
−3	2	1	1	0	1	1	0	0	1	1	0
−2	−1	−1	0	−1	0	−1	0	0	0	−1	0
−1	1	0	1	−1	0	0	1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	1	0	0	1	0	0	0
2	1	1	0	1	1	0	0	1	0	0	0
3	2	1	1	0	1	1	0	1	1	0	0
4	3	1	2	−1	1	2	0	1	2	0	0
5	5	2	3	−1	1	3	1	1	3	1	0
6	8	4	4	0	1	4	3	1	4	3	0
7	13	7	6	1	2	5	6	1	5	6	1
8	21	11	10	1	5	6	10	1	6	10	4
9	34	17	17	0	11	8	15	2	7	15	10
10	55	27	28	−1	21	13	21	6	8	21	20
11	89	44	45	−1	36	24	29	16	10	28	35
12	144	72	72	0	57	45	42	36	16	36	56
13	233	117	116	1	86	81	66	71	32	46	84
14	377	189	188	1	128	138	111	127	68	62	120
15	610	305	305	0	194	224	192	211	139	94	166
16	987	493	494	−1	305	352	330	331	266	162	228
17	1597	798	799	−1	497	546	554	497	477	301	322

TABLE 2. Numbers $L_n^{(m,k)}$

n	F_n	$m=2$, $k=0$	$m=2$, $k=1$	δ'_n	$m=3$, $k=0$	$m=3$, $k=1$	$m=3$, $k=2$	$m=4$, $k=0$	$m=4$, $k=1$	$m=4$, $k=2$	$m=4$, $k=3$
0	2	2	0	2	2	0	0	2	0	0	0
1	1	1	0	1	1	0	0	1	0	0	0
2	3	1	2	−1	1	2	0	1	2	0	0
3	4	1	3	−2	1	3	0	1	3	0	0
4	7	3	4	−1	1	4	2	1	4	2	0
5	11	6	5	1	1	5	5	1	5	5	0
6	18	10	8	2	3	6	9	1	6	9	2
7	29	15	14	1	8	7	14	1	7	14	7
8	47	23	24	−1	17	10	20	3	8	20	16
9	76	37	39	−2	31	18	27	10	9	27	30
10	123	61	62	−1	51	35	37	26	12	35	50
11	199	100	99	1	78	66	55	56	22	44	77
12	322	162	160	2	115	117	90	106	48	56	112
13	521	261	260	1	170	195	156	183	104	78	156
14	843	421	422	−1	260	310	273	295	210	126	212
15	1364	681	683	−2	416	480	468	451	393	230	290

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