GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS AND THEIR ASSOCIATED DIAGONAL POLYNOMIALS

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1. INTRODUCTION

Horadam [7], in a recent article, defined two sequences of polynomials $J_n(x)$ and $j_n(x)$, the Jacobsthal and Jacobsthal-Lucas polynomials, respectively, and studied their properties. In the same article, he also defined and studied the properties of the rising and descending polynomials $R_n(x)$, $r_n(x)$, $D_n(x)$, and $d_n(x)$, which are fashioned in a manner similar to those for Chebyshev, Fermat, and other polynomials (see [2], [3], [4], [5], and [6]).

The purpose of this article is to extend these results to the generalized Fibonacci and Lucas polynomials defined by

$$U_n(x, y) = xU_{n-1}(x, y) + yU_{n-2}(x, y) \quad (n \geq 2), \tag{1.1a}$$

with

$$U_0(x, y) = 0, \quad U_1(x, y) = 1, \tag{1.1b}$$

and

$$V_n(x, y) = xV_{n-1}(x, y) + yV_{n-2}(x, y) \quad (n \geq 2), \tag{1.2a}$$

with

$$V_0(x, y) = 2, \quad V_1(x, y) = x. \tag{1.2b}$$

In Section 2, we will give some basic properties of the polynomials $U_n(x, y)$ and $V_n(x, y)$, most of which are generalizations of those given in [7] for $J_n(x)$ and $j_n(x)$. In Section 3, we will derive some new properties of $U_n(x, y)$ and $V_n(x, y)$ concerning their derivatives, as well as the differential equations they satisfy. In the remaining sections, we will define and study the properties of the rising and descending diagonal polynomials associated with $U_n(x, y)$ and $V_n(x, y)$, thus generalizing the results already known for Fibonacci, Lucas, Chebyshev, Fermat, and Jacobsthal polynomials.

2. BASIC PROPERTIES OF $U_n(x, y)$ AND $V_n(x, y)$

Binet Forms:

$$U_n(x, y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.1}$$

$$V_n(x, y) = \alpha^n + \beta^n, \tag{2.2}$$

where

$$\alpha + \beta = x, \quad \alpha\beta = -y, \tag{2.3a}$$

$$\alpha - \beta = \sqrt{\Delta}, \quad \Delta = x^2 + 4y, \tag{2.3b}$$

$$2\alpha = x + \sqrt{\Delta}, \quad 2\beta = x - \sqrt{\Delta}. \tag{2.3c}$$

Simson Formulas:

$$U_{n+1}(x, y)U_{n-1}(x, y) - U_n^2(x, y) = (-1)^n y^{n-1}, \tag{2.4}$$

$$V_{n+1}(x, y)V_{n-1}(x, y) - V_n^2(x, y) = (-1)^n y^{n-1}\Delta. \tag{2.5}$$

Summation Formulas:

$$\sum_{0}^{n} U_i(x, y) = \frac{1}{x+y-1}[U_{n+1}(x, y) + yU_n(x, y) - 1], \tag{2.6}$$

$$\sum_{0}^{n} V_i(x, y) = \frac{1}{x+y-1}[V_{n+1}(x, y) + yV_n(x, y) + (x-2)]. \tag{2.7}$$

Important Interrelations:

$$V_n(x, y) = U_{n+1}(x, y) + yU_{n-1}(x, y), \tag{2.8}$$

$$V_n(x, y) + xU_n(x, y) = 2U_{n+1}(x, y), \tag{2.9}$$

$$V_n(x, y) - xU_n(x, y) = 2yU_{n-1}(x, y), \tag{2.10}$$

$$\Delta U_n(x, y) = V_{n+1}(x, y) + yV_{n-1}(x, y), \tag{2.11}$$

$$\Delta U_n(x, y) = 2V_{n+1}(x, y) - xV_n(x, y), \tag{2.12}$$

$$U_{2n}(x, y) = U_n(x, y)V_n(x, y), \tag{2.13}$$

$$V_{2n}(x, y) = V_n^2(x, y) - 2(-y)^n, \tag{2.14}$$

$$V_{2n}(x, y) = \Delta U_n^2(x, y) + 2(-y)^n, \tag{2.15}$$

$$\Delta U_n^2(x, y) + V_n^2(x, y) = 2V_{2n}(x, y), \tag{2.16}$$

$$2U_{m+n}(x, y) = U_m(x, y)V_n(x, y) + V_m(x, y)U_n(x, y), \tag{2.17}$$

$$2V_{m+n}(x, y) = V_m(x, y)V_n(x, y) + \Delta U_m(x, y)U_n(x, y). \tag{2.18}$$

All the above results from (2.4)-(2.18) may be derived using the Binet forms (2.1) and (2.2) or, alternately, using the earlier results of Horadam [8]. Most of these results are to be found in Lucas ([10], Ch. 18). Now we let $X = \alpha$ and $Y = \beta$ in the following identities, where α and β are given by (2.3), X and Y arbitrary:

$$\frac{X^n - Y^n}{X - Y} = \sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-r-1}{r} (XY)^r (X+Y)^{n-2r-1} \quad (n \geq 0), \tag{2.19}$$

$$X^n + Y^n = \sum_{r=0}^{[n/2]} (-1)^r \frac{n}{n-r} \binom{n-r}{r} (XY)^r (X+Y)^{n-2r} \quad (n > 0). \tag{2.20}$$

We can then easily establish the following expressions for $U_n(x, y)$ and $V_n(x, y)$.

Closed Form Expressions:

$$U_n(x, y) = \sum_{r=0}^{[(n-1)/2]} \binom{n-r-1}{r} x^{n-2r-1} y^r, \tag{2.21}$$

$$V_n(x, y) = \sum_{r=0}^{[n/2]} \frac{n}{n-r} \binom{n-r}{r} x^{n-2r} y^r \quad (n > 0). \tag{2.22}$$

It is seen from (2.21) and (2.22) that $U_{2n}(x, y)$ and $V_{2n-1}(x, y)$ are odd polynomials in x of degree $(2n-1)$ and polynomials in y of degree $(n-1)$, while $U_{2n+1}(x, y)$ and $V_{2n}(x, y)$ are even polynomials in x of degree $2n$ and polynomials in y of degree n. It may be mentioned that expression (2.21) for $U_n(x, y)$ has already been established by Hoggatt and Long [3]; however, the expression for $V_n(x, y)$ is new. By letting $x = 1$ and $y = 2x$, we obtain the results of Horadam [7] for the polynomials $J_n(x)$ and $j_n(x)$.

Hoggatt and Long [3] have shown that

$$U_n(x, y) = \prod_{k=1}^{n-1}\left\{x - 2\sqrt{-y}\cos\left(\frac{k}{n}\pi\right)\right\} \quad (n \geq 2). \tag{2.23}$$

Using a similar procedure, or by using the technique used by Swamy [11] in obtaining the zeros of Morgan-Voyce polynomials, we can show that

$$V_n(x, y) = \prod_{k=1}^{n}\left\{x - 2\sqrt{-y}\cos\left(\frac{2k-1}{2n}\pi\right)\right\} \quad (n \geq 2). \tag{2.24}$$

We may now rewrite expressions (2.23) and (2.24) to express the polynomials $U_n(x, y)$ and $V_n(x, y)$ in the product form.

Product Form:

$$U_n(x, y) = x^{\delta_n} \prod_{k=1}^{[(n-1)/2]}\left\{x^2 + 4y\cos^2\left(\frac{k}{n}\pi\right)\right\} \quad (n > 2), \tag{2.25}$$

$$V_n(x, y) = x^{1-\delta_n} \prod_{k=1}^{[n/2]}\left\{x^2 + 4y\cos^2\left(\frac{2k-1}{2n}\pi\right)\right\} \quad (n \geq 2), \tag{2.26}$$

where

$$\delta_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2.27}$$

By letting $x = 1$ and $y = 2x$ in (2.26) and (2.27), we get the zeros of the Jacobsthal polynomials $J_n(x)$ and $j_n(x)$ to be, respectively,

$$x = -\frac{1}{8}\sec^2\left(\frac{k}{n}\pi\right), \quad k = 1, 2, \ldots, (n-1), \tag{2.28}$$

and

$$x = -\frac{1}{8}\sec^2\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n. \tag{2.29}$$

The generating functions for $U_n(x, y)$ and $V_n(x, y)$ are given below.

Generating Functions:

$$U(x, y, t) = \sum_{i=1}^{\infty} U_i(x, y)t^{i-1} = \{1 - t(x + yt)\}^{-1}, \tag{2.30}$$

$$V(x, y, t) = \sum_{i=1}^{\infty} V_i(x, y)t^{i} = (2 - xt)\{1 - t(x + yt)\}^{-1} \tag{2.31}$$

$$= 1 + (1 + yt^2)\{1 - t(x + yt)\}^{-1} \tag{2.32}$$

3. DERIVATIVE PROPERTIES

From (2.30), (2.31), and (2.32), a number of relations involving the derivatives of $U_n(x, y)$ and $V_n(x, y)$ may be derived. However, only the following derivative relations are listed here. Throughout this section, where not explicitly mentioned, U, V, U_n, and V_n stand for $U(x, y, t)$, $V(x, y, t)$, $U_n(x, y)$, and $V_n(x, y)$, respectively. We can prove that

$$\frac{\partial V}{\partial x}=t\frac{\partial}{\partial t}(tU), \tag{3.1}$$

$$\frac{\partial V}{\partial y}=t\frac{\partial}{\partial t}(t^2U), \tag{3.2}$$

$$\frac{\partial U}{\partial y}=t\frac{\partial U}{\partial x}, \tag{3.3}$$

$$\frac{\partial V}{\partial y}=t\frac{\partial V}{\partial x}+t^2U, \tag{3.4}$$

$$x\frac{\partial U}{\partial x}+2y\frac{\partial U}{\partial y}=t\frac{\partial U}{\partial t}, \tag{3.5}$$

$$x\frac{\partial V}{\partial x}+2y\frac{\partial V}{\partial y}=t\frac{\partial V}{\partial t}. \tag{3.6}$$

The above results are now established. From the generating function (2.30), we have

$$\frac{1}{t}\frac{\partial U}{\partial x}=\frac{1}{t^2}\frac{\partial U}{\partial y}=\frac{1}{x+2yt}\frac{\partial U}{\partial t}=U^2. \tag{3.7}$$

We see that (3.3) and (3.5) follow directly from (3.7). Now, from (2.32) and (2.31), we have

$$\frac{\partial V}{\partial x}=t(1+yt^2)U^2, \tag{3.8a}$$

$$\frac{\partial V}{\partial y}=t^2(2-xt)U^2. \tag{3.8b}$$

However,

$$\frac{\partial}{\partial t}(tU)=(1+yt^2)U^2, \tag{3.9a}$$

$$\frac{\partial}{\partial t}(t^2U)=t(2-xt)U^2. \tag{3.9b}$$

Relation (3.1) follows directly from (3.8a) and (3.9a), while (3.2) follows directly from (3.8b) and (3.9b). Also, from (3.8a) and (3.8b), we have

$$x\frac{\partial V}{\partial x}+2y\frac{\partial V}{\partial y}=t(x-xyt^2+4yt)U^2=t\frac{\partial V}{\partial t},$$

thus establishing (3.6). Finally, we have, from (3.8a),

$$t^2U+t\frac{\partial V}{\partial x}=t^2(1-xt-yt^2)U^2+t^2(1+yt^2)U^2$$

$$=t^2(2-xt)U^2=\frac{\partial V}{\partial y},\ \text{using (3.8b)}.$$

Thus, relation (3.4) is established. Using the above relations (3.1) to (3.6) and the generating functions for $U(x,y,t)$ and $V(x,y,t)$ given by (2.30) to (2.32), we can obtain the following relationships, where the primes indicate partial derivatives with respect to x and dots those with respect to y:

$$V_n'(x,y)=nU_n(x,y),\ \text{using (3.1), (2.30), and (2.31)}, \tag{3.10}$$

$$\dot{V}_n(x,y)=nU_{n-1}(x,y),\ \text{using (3.2), (2.3), and (2.31)}, \tag{3.11}$$

$$\dot{U}_{n+1}(x,y) = U_n'(x,y), \text{ using (3.3) and (2.30), or from (3.10) and (3.11),} \tag{3.12}$$

$$n\dot{V}_{n+1}(x,y) = (n+1)V_n'(x,y), \text{ using (3.10) and (3.11),} \tag{3.13}$$

$$\dot{V}_{n+1}(x,y) = V_n'(x,y) + U_n(x,y), \text{ using (3.4), (2.30), and (2.31), or from (3.10) and (3.13),} \tag{3.14}$$

$$xU_n'(x,y) + 2y\dot{U}_n(x,y) = (n-1)U_n(x,y), \text{ using (3.5) and (2.30),} \tag{3.15}$$

$$xV_n'(x,y) + 2y\dot{V}_n(x,y) = nV_n(x,y), \text{ using (3.6) and (2.31).} \tag{3.16}$$

We shall illustrate the procedure for proving the above results by establishing (3.12) and (3.16); the other results may be established in a similar manner. Substituting (2.30) and (2.31) in (3.3) and (3.6), respectively, we get:

$$\sum_1^\infty U_i'(x,y)t^{i-1} = t\sum_1^\infty \dot{U}_i(x,y)t^{i-1};$$

$$x\sum_0^\infty V_i'(x,y)t^i + 2y\sum_0^\infty \dot{V}_i(x,y)t^i = t\sum_1^3 iV_i(x,y)t^{i-1}.$$

Comparing the coefficients of like powers of t on both sides of the above equations, we obtain (3.12) and (3.16), respectively.

Using the results of (3.12) and (3.13), we may now derive the following relations for the higher-order derivatives of $U_n(x,y)$ and $V_n(x,y)$, where $D_x^{(r)}$ and $D_y^{(r)}$ denote the derivatives with respect to x and y, respectively.

$$D_y^{(r)}U_{n+1}(x,y) = D_x^{(r)}U_{n-r+1}(x,y), \tag{3.17}$$

$$(n-r+1)D_y^{(r)}V_{n+1}(x,y) = (n+1)D_x^{(r)}V_{n-r+1}(x,y). \tag{3.18}$$

We will now derive the linear differential equations satisfied by $U_n(x,y)$ and $V_n(x,y)$. From (2.12), we have $\Delta\dot{U}_{n-1} + 4U_{n-1} = 2\dot{V}_n - x\dot{V}_{n-1}$. Hence,

$$\begin{aligned}\frac{1}{n}\Delta\ddot{V}_n + 4U_{n-1} &= 2nU_{n-1} - x(n-1)U_{n-2}, \text{ using (3.11),}\\ &= 2nU_{n-1} - (n-1)[2U_{n-1} - V_{n-2}], \text{ using (2.9)}\\ &= 2U_{n-1} + (n-1)(\Delta U_{n-1} - V_n)/y, \text{ using (2.11).}\end{aligned}$$

Therefore,

$$y\Delta\ddot{V}_n + \{2y - (n-1)\Delta\}nU_{n-1} + n(n-1)V_n = 0. \tag{3.19}$$

Substituting (3.11) in (3.19), we see that $V_n(x,y) = z$ satisfies the differential equation given by

$$y(x^2+4y)\ddot{z} + \{2y - (n-1)(x^2+4y)\}\dot{z} + n(n-1)z = 0. \tag{3.20}$$

Differentiating (3.19) again with respect to y and again making use of the result (3.11), we get

$$y(x^2+4y)n\ddot{U}_{n-1} + \{6y - (n-2)(x^2+4y)\}n\dot{U}_{n-1} + (n-2)(n-3)nU_{n-1} = 0.$$

Now, replacing $(n-1)$ by n, we see that $U_n(x,y) = z$ satisfies the differential equation

$$y(x^2+4y)\ddot{z} + \{6y - (n-1)(x^2+4y)\}\dot{z} + (n-1)(n-2)z = 0. \tag{3.21}$$

Since the Jacobsthal polynomials [7] $J_n(x)$ and $j_n(x)$ are given by

$$J_n(x) = U_n(1, 2x) \quad \text{and} \quad j_n(x) = V_n(1, 2x), \tag{3.22}$$

we see, from (3.21), that $J_n(x)$ satisfies the differential equation

$$x(8x+1)z'' - \{4(2n-5)x + (n-1)\}z' + 2(n-1)(n-2)z = 0, \tag{3.23}$$

while, from (3.20), we see that $j_n(x)$ satisfies the equation

$$x(8x+1)z'' - \{4(2n-3)x + (n-1)\}z' + 2n(n-1)z = 0. \tag{3.24}$$

Recall that $y = 2x$ implies that $\dot{z} = \frac{1}{2}z'$ and $\ddot{z} = \frac{1}{4}z''$.

In a similar way, differentiating both sides of (2.11) with respect to x, and utilizing (3.10), (2.8), and (1.1a), we can show that $V_n(x, y)$ satisfies the equation

$$(x^2 + 4y)z'' + xz' - n^2 z = 0. \tag{3.25}$$

Differentiating (3.25) with respect to x once and making use of (3.10), we see that $U_n(x, y)$ satisfies the equation

$$(x^2 + 4y)z'' + 3xz' + (1 - n^2)z = 0. \tag{3.26}$$

It should be noted that equations (3.25) and (3.26) appear in [1] as equations (1.11) and (2.6), respectively, in a slightly varied notation. It should also be noted that, after the submission of this article, an article by Horadam [9] appeared generalizing the results given in (3.20), (3.21), (3.25), and (3.26).

Also, from (3.11) and (1.2), we can show that

$$\dot{V}_n(x, y) = \frac{n}{n-1} x\dot{V}_{n-1}(x, y) + \frac{n}{n-2} y\dot{V}_{n-2}(x, y) \quad (n \geq 3), \tag{3.27a}$$

while, from (3.10) and (1.2), we can prove that

$$V_n'(x, y) = \frac{n}{n-1} xV_{n-1}'(x, y) + \frac{n}{n-2} yV_{n-2}'(x, y) \quad (n \geq 3). \tag{3.28a}$$

Thus, we see that both $\dot{V}_n(x, y)$ and $V_n'(x, y)$ satisfy the same recurrence relation, but with different initial conditions as given by

$$\dot{V}_1(x, y) = 0, \quad \dot{V}_2(x, y) = 2, \tag{3.27b}$$

$$V_i'(x, y) = 1, \quad V_2'(x, y) = 2x. \tag{3.28b}$$

4. RISING DIAGONAL POLYNOMIALS

Let us first consider the rising diagonal polynomials $R_n(x, y)$ associated with $U_n(x, y)$; these polynomials are formed the same way as the rising diagonal polynomials associated with Fermat, Chebyshev, Jacobsthal, and other similar polynomials (see [2], [4], [5], [6], and [7]). Thus, from (2.21), we see that $R_0(x, y) = 0$, $R_1(x, y) = 1$, $R_2(x, y) = x, \ldots,$

$$R_n(x, y) = x^{n-1} + \binom{n-3}{1} x^{n-4} y + \binom{n-5}{2} x^{n-7} y^2 + \binom{n-7}{3} x^{n-10} y^3 + \cdots.$$

The above may be rewritten as

$$R_n(x, y) = x^{n-1} + \binom{n-1-2\cdot 1}{1} x^{n-1-3\cdot 1} y + \binom{n-1-2\cdot 2}{2} x^{n-1-3\cdot 2} y^2 +$$

$$+\binom{n-1-2\cdot 3}{3}x^{n-1-3\cdot 3}y^3+\cdots+\binom{n-1-2\cdot[\frac{n-1}{3}]}{[\frac{n-1}{3}]}y^{[\frac{n-1}{3}]}.$$

Hence,

$$R_n(x,y)=\sum_{i=0}^{[(n-1)/3]}\binom{n-1-2i}{i}x^{n-1-3i}y^i \quad (n\geq 1),\quad R_0(x,y)=0. \tag{4.1}$$

Similarly, starting with (2.22), we may show that the rising polynomials $r_n(x,y)$ associated with $V_n(x,y)$ are given by

$$r_n(x,y)=\sum_{i=0}^{[n/3]}\frac{n-i}{n-2i}\binom{n-2i}{i}x^{n-3i}y^i \quad (n\geq 1),\quad r_0(x,y)=2. \tag{4.2}$$

We now derive some interesting relationships for these rising polynomials including the recurrence relations. From (4.1) and (4.2), we have

$$\begin{aligned}r_n(x,y)+xR_n(x,y)&=\sum_{i=0}^{[n/3]}\frac{n-i}{n-2i}\binom{n-2i}{i}x^{n-3i}y^i+\sum_{i=0}^{[(n-1)/3]}\binom{n-1-2i}{i}x^{n-3i}y^i\\&=2\sum_{i=0}^{[n/3]}\binom{n-2i}{i}x^{n-3i}y^i=2R_{n+1}(x,y).\end{aligned}$$

Hence,

$$r_n(x,y)+xR_n(x,y)=2R_{n+1}(x,y). \tag{4.3}$$

Similarly, we can show that

$$r_n(x,y)-xR_n(x,y)=2yR_{n-2}(x,y) \quad (n\geq 2). \tag{4.4}$$

Hence, from (4.3) and (4.4),

$$\begin{aligned}r_n(x,y)&=R_{n+1}(x,y)+yR_{n-2}(x,y) \quad (n\geq 2),\\R_{n+1}(x,y)&=xR_n(x,y)+yR_{n-2}(x,y) \quad (n\geq 2).\end{aligned} \tag{4.5}$$

Thus, we see that $R_n(x,y)$ satisfies the recurrence relation

$$R_n(x,y)=xR_{n-1}(x,y)+yR_{n-3}(x,y) \quad (n\geq 3), \tag{4.6a}$$

with

$$R_0(x,y)=0,\quad R_1(x,y)=1,\quad R_2(x,y)=x. \tag{4.6b}$$

Similarly, using (4.3), (4.4), and (4.5), we can deduce that $r_n(x,y)$ satisfies the recurrence relation

$$r_n(x,y)=xr_{n-1}(x,y)+yr_{n-3}(x,y) \quad (n\geq 3), \tag{4.7a}$$

with

$$r_0(x,y)=2,\quad r_1(x,y)=x,\quad r_2(x,y)=x^2. \tag{4.7b}$$

It is interesting to compare the relations (4.6), (4.7), (4.5), (4.3), and (4.4) with their counterparts for $U_n(x,y)$ and $V_n(x,y)$ given, respectively, by (1.1), (1.2), (2.8), (2.9), and (2.10).

The generating functions for $R_n(x,y)$ and $r_n(x,y)$ may be found by following the usual technique. They are given by

$$R(x,y,t)=\sum_{i=1}^{\infty}R_i(x,y)t^{i-1}=\{1-t(x+yt^2)\}^{-1}, \tag{4.8}$$

$$r(x,y,t)=\sum_{i=0}^{\infty}r_i(x,y)t^i=(2-xt)\{1-t(x+yt^2)\}^{-1} \tag{4.9}$$

$$=1+(1+yt^3)\{1-t(x+yt^2)\}^{-1}. \tag{4.10}$$

Using these generating functions, we may now derive a number of results concerning the derivatives of $R(x,y,t)$ and $r(x,y,t)$ where, for the sake of convenience, R and r are used for $R(x,y,t)$ and $r(x,y,t)$. A few of these results are:

$$\frac{\partial R}{\partial y}=t^2\frac{\partial R}{\partial x}, \tag{4.11}$$

$$\frac{\partial r}{\partial y}=t^2\frac{\partial r}{\partial x}+t^3R, \tag{4.12}$$

$$x\frac{\partial R}{\partial x}+3y\frac{\partial R}{\partial y}=t\frac{\partial R}{\partial t}, \tag{4.13}$$

$$x\frac{\partial r}{\partial x}+3y\frac{\partial r}{\partial y}=t\frac{\partial r}{\partial t}. \tag{4.14}$$

The above results may be established in a way similar to those given in (3.1) to (3.6). From the above results, we may derive the following relationships for the derivatives of $R_n(x,y)$ and $r_n(x,y)$, where again the primes indicate partial derivatives with respect to x and dots those with respect to y:

$$\dot{R}_{n+2}(x,y)=R_n'(x,y), \tag{4.15}$$

$$\dot{r}_{n+2}(x,y)=r_n'(x,y)+R_n(x,y), \tag{4.16}$$

$$xR_n'(x,y)+3y\dot{R}_n(x,y)=(n-1)R_n(x,y) \tag{4.17}$$

$$xr_n'(x,y)+3y\dot{r}_n(x,y)=nr_n(x,y). \tag{4.18}$$

Again, it is interesting to compare the relationships (4.11), (4.12), (4.13), (4.14), (4.15), (4.16), (4.17), and (4.18) with their counterparts for $U_n(x,y)$ and $V_n(x,y)$, namely, the relations (3.3), (3.4), (3.5), (3.6), (3.12), (3.14), (3.15), and (3.16), respectively.

5. DESCENDING DIAGONAL POLYNOMIALS

Let us now consider the descending diagonal polynomials $D_n(x,y)$ and $d_n(x,y)$ associated with the polynomials $U_n(x,y)$ and $V_n(x,y)$, respectively; these are formed the same way as the descending diagonal functions associated with Chebyshev, Fermat, Jacobsthal, and other similar polynomials (see [2], [4], [5], [6], and [7]). Thus, from (2.21), we see that the descending polynomial $D_n(x,y)$ associated with $U_n(x,y)$ is given by

$$D_0(x,y)=0,\quad D_1(x,y)=1,\quad D_2(x,y)=x+y,\ldots,$$

$$D_n(x,y)=\binom{n-1}{0}x^{n-1}+\binom{n-1}{1}x^{n-2}y+\cdots+\binom{n-1}{n-1}y^{n-1}=(x+y)^{n-1}.$$

Hence,

$$D_n(x,y)=\sum_{i=0}^{n-1}\binom{n-1}{i}x^{n-1-i}y^i=(x+y)^{n-1}\quad(n\geq 1),\quad D_0(x,y)=0. \tag{5.1}$$

Similarly, starting with relation (2.22), we can obtain the descending polynomial $d_n(x, y)$ associated with the polynomial $V_n(x, y)$ to be

$$d_n(x,y)=\sum_{i=0}^{n}\frac{n+i}{n}\binom{n}{i}x^{n-i}y^i \quad (n\geq 1),\quad d_0(x,y)=2. \tag{5.2}$$

Now consider

$$D_{n+1}(x,y)+yD_n(x,y)=\sum_{i=0}^{n}\left\{\binom{n}{i}+\binom{n-1}{i-1}\right\}x^{n-i}y^i=\sum_{i=0}^{n}\frac{n+i}{n}\binom{n}{i}x^{n-i}y^i \quad (n\geq 1).$$

Hence

$$d_n(x,y)=D_{n+1}(x,y)+yD_n(x,y) \quad (n\geq 1). \tag{5.3}$$

Thus,

$$d_n(x,y)=(x+2y)(x+y)^{n-1} \quad (n\geq 1). \tag{5.4}$$

We also have, from (5.1) and (5.4),

$$\frac{D_{n+1}(x,y)}{D_n(x,y)}=\frac{d_{n+1}(x,y)}{d_n(x,y)}=x+y \quad (n\geq 1), \tag{5.5}$$

$$d_{n+1}(x,y)+yd_n(x,y)=(x+2y)^2D_n(x,y) \quad (n\geq 1). \tag{5.6}$$

We may also formulate the following generating functions for the descending polynomials $D_n(x, y)$ and $d_n(x, y)$ by following the usual procedure:

$$D(x,y,t)=\sum_{i=1}^{\infty}D_i(x,y)t^{i-1}=\{1-(x+y)t\}^{-1}, \tag{5.7}$$

$$d(x,y,t)=\sum_{i=1}^{\infty}d_i(x,y)t^{i-1}=(x+2y)\{1-(x+y)t\}^{-1}. \tag{5.8}$$

From the above generating functions, we may deduce the following relations for the derivatives of $D(x,y,t)$ and $d(x,y,t)$ where, for the sake of convenience, D and d are used for $D(x,y,t)$ and $d(x,y,t)$:

$$\frac{\partial D}{\partial y}=\frac{\partial D}{\partial x}, \tag{5.9}$$

$$\frac{\partial d}{\partial y}=\frac{\partial d}{\partial x}+D, \tag{5.10}$$

$$x\frac{\partial D}{\partial x}+y\frac{\partial D}{\partial y}=t\frac{\partial D}{\partial t}, \tag{5.11}$$

$$x\frac{\partial d}{\partial x}+y\frac{\partial d}{\partial y}=t\frac{\partial d}{\partial t}+d, \tag{5.12}$$

$$(x+y)\frac{\partial D}{\partial y}=(x+y)\frac{\partial D}{\partial x}=t\frac{\partial D}{\partial t}, \tag{5.13}$$

$$(x+y)\frac{\partial d}{\partial y}x=t\frac{\partial d}{\partial t}+(x+y)D, \tag{5.14}$$

$$(x+y)\frac{\partial d}{\partial y}=t\frac{\partial d}{\partial t}+2(x+y)D. \tag{5.15}$$

Using the above relations, we may write the corresponding interrelations for the derivatives of the polynomials $D_n(x, y)$ and $d_n(x, y)$ with respect to x and y as has been done for $R_n(x, y)$ and $r_n(x, y)$.

6. CONCLUDING REMARKS

We have generalized all the known results concerning the diagonal functions associated with Fibonacci, Lucas, Chebyshev, Fermat, Pell, and Jacobsthal polynomials to the case of diagonal functions associated with the generalized polynomials given by (1.1) and (1.2). We have also derived a number of new interesting results concerning the derivatives of $U_n(x, y)$ and $V_n(x, y)$ with respect to y, the differential equations satisfied by these polynomials, as well as the interrelations between their derivatives with respect to x and y. Similar results have also been derived for both the rising and the descending diagonal polynomials associated with $U_n(x, y)$ and $V_n(x, y)$; however, we have not been able to find the differential equations satisfied by $R_n(x, y)$, $r_n(x, y)$, and $d_n(x, y)$ with respect to either x or y. It may also be observed that the descending (rising) polynomials associated with the rising (descending) polynomials of $U_n(x, y)$ and $V_n(x, y)$ are, respectively, $U_n(x, y)$ and $V_n(x, y)$. This answers one of the questions raised by Horadam [7] regarding the rising polynomials of the descending polynomials of $J_n(x)$ and $j_n(x)$ as well as the descending polynomials of the rising polynomials of $J_n(x)$ and $j_n(x)$.

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