NOTE ON THE PIERCE EXPANSION OF A LOGARITHM

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1. INTRODUCTION

Daniel Shanks, in [10], introduced a very interesting algorithm for obtaining the regular continued fraction expansion of the logarithm of a number. A concise presentation can be found in [5]; however, it is difficult to find in the literature (e.g., texts such as [1], [2], [4], and [11] do not mention it). Shanks' algorithm is the inspiration for the one we present here using Pierce expansions, but it can be adapted easily to other well-known expansions such as Engel's or Sylvester's (see [3] and [6] for details).

A very brief description of Pierce expansions is given below. A more complete account can be found in [7], [8], [9], and [12].

Definition 1: The Pierce expansion of a real number $\alpha \in (0, 1]$ is an expression of the form

$$\frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n-1}}{a_1 a_2 \ldots a_{n-1} a_n} + \cdots, \tag{1}$$

where $a_1, a_2, \ldots, a_n, \ldots$ constitute a strictly increasing sequence of positive integers. In the case that the sum above is finite, we call it a *terminating* (or *finite*) expansion and then we add the condition that the last two terms, a_{n-1} and a_n, are not consecutive: $a_{n-1} < a_n - 1$.

We denote (1) by $\langle a_1, a_2, \ldots, a_n, \ldots \rangle$. The requirement that $a_{n-1} < a_n - 1$ is to ensure uniqueness in the case of terminating expansions as $\langle a, \ldots, k \rangle = \langle a, \ldots, k, k+1 \rangle$.

Pierce expansions constitute a system of representation of real numbers in (0, 1], as the following theorem proves.

Theorem 1: Any real number α, $0 < \alpha \leq 1$, has a unique representation as a Pierce expansion: rationals as finite expansions and irrationals as infinite expansions.

We include a sketch of the proof as a guide for the algorithm of the next section.

Proof: Uniqueness is the result of observing that a Pierce expansion verifies

$$\frac{1}{a_1 + 1} < \langle a_1, a_2, \ldots, a_n, \ldots \rangle \leq \frac{1}{a_1}.$$

The existence is easily justified by the following algorithm. If $x \in (0, 1]$, its Pierce expansion, $\langle a_1, a_2, \dots, a_n, \dots \rangle$, is obtained as follows:

Step 1. $x_0 \leftarrow x$; $i \leftarrow 1$.

Step 2. $a_i = \lfloor 1/x_{i-1} \rfloor$; $x_i \leftarrow 1 - a_i x_{i-1}$.

Step 3. If $x_i = 0$, then stop; else $i \leftarrow i+1$ and go to Step 2.

If x is a rational number p/q, the algorithm will eventually terminate as Step 2 requires, on the first iteration, that we perform the division of q by p and after that the division of q by the successive remainders which are obviously decreasing and eventually must become 0. In that case, the expansion will be finite. If x is not rational, the algorithm will never terminate but will provide a series that is easily seen to converge to x. □

2. THE PIERCE EXPANSION OF A LOGARITHM

The following algorithm will provide us with the Pierce expansion of the logarithm of a number in base $b > 1$. It is easily extended to bases < 1. Let x be a real number, $1 < x < b$. Our aim is to find $\log_b x = \langle a_1, \dots, a_n, \dots \rangle$, where the Pierce expansion can be terminating or not.

Let $x_1 = x$. If we have $1 < x_i < b$, let a_i be the sole positive integer verifying

$$x_i^{a_i} \le b < x_i^{a_i+1}. \tag{2}$$

The integer a_i is well defined as $1 < x_i < b$ and the sequence $\{x_i^n\}_{n \in \mathbb{N}}$ is strictly increasing. Now we define

$$x_{i+1} = \frac{b}{x_i^{a_i}}. \tag{3}$$

From (2), we immediately have $1 \le x_{i+1} < x_i < b$. If $x_{i+1} > 1$, we can continue. Let us suppose we have reached an x_{n+1} such that $1 \le x_{n+1} < x_n < \cdots < x_1 < b$.

Lemma 1: For all i $(1 \le i \le n)$,

$$x = b^{\left(\frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{i-1}}{a_1 a_2 \dots a_i}\right)} \cdot x_{i+1}^{\frac{(-1)^i}{a_1 \dots a_i}}. \tag{4}$$

Proof: We shall proceed by induction on i. For $i = 1$ we have, from (3),

$$x_1^{a_1} = b \cdot x_2^{-1} \Rightarrow x = x_1 = b^{\frac{1}{a_1}} \cdot x_2^{-\frac{1}{a_1}},$$

therefore (4) is verified. If we assume it is verified until $i = k-1$, from the definition of x_{k+1} we have

$$x_{k+1} = \frac{b}{x_k^{a_k}} \Rightarrow x_k = (b \cdot x_{k+1}^{-1})^{\frac{1}{a_k}} = b^{\frac{1}{a_k}} \cdot x_{k+1}^{-\frac{1}{a_k}}.$$

By the induction hypothesis,

$$x = b^{\left(\frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{k-2}}{a_1 a_2 \dots a_{k-1}}\right)} \cdot x_k^{\frac{(-1)^{k-1}}{a_1 \dots a_{k-1}}},$$

and, replacing x_k by its value, we have

$$x = b^{\left(\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{k-2}}{a_1a_2\ldots a_{k-1}}\right)} \cdot b^{\frac{(-1)^{k-1}}{a_1\ldots a_{k-1}a_k}} \cdot x_k^{\frac{(-1)^{k-1}}{a_1\ldots a_{k-1}}}$$
$$= b^{\left(\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{k-1}}{a_1a_2\ldots a_k}\right)} \cdot x_{k+1}^{\frac{(-1)^k}{a_1\ldots a_k}}.$$

This completes the proof of Lemma 1. □

Now, if $x_{n+1}=1$, by the former lemma,

$$x = b^{\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{n-1}}{a_1a_2\ldots a_n}},$$

and we shall be done as soon as we prove $a_1 < a_2 < \cdots < a_{n-1} < a_n - 1$, to which we turn at once.

In general, if $1 \le k < n-1$, from the definition of a_k,

$$x_k^{a_k} < b < x_k^{a_k+1},$$

there exists a real number σ, $a_k < \sigma < a_k+1$, such that $b = x_k^{\sigma}$. Consequently, we have

$$\frac{1}{a_k+1} < \frac{1}{\sigma} < \frac{1}{a_k} \quad \text{and} \quad x_k = b^{\frac{1}{\sigma}}. \tag{5}$$

Now, the first inequality in (5) implies the existence of a real number α, $\alpha > a_k + 1$, such that

$$\frac{1}{\sigma} = \frac{1}{a_k} - \frac{1}{a_k\alpha}. \tag{6}$$

We can write

$$x_k = b^{\frac{1}{\sigma}} = b^{\frac{1}{a_k}-\frac{1}{a_k\alpha}} \Rightarrow x_k^{a_k} \cdot b^{\frac{1}{\alpha}} = b;$$

therefore, from the definition of x_{k+1}, we can also write

$$x_{k+1} = \frac{b}{x_k^{a_k}} = b^{\frac{1}{\alpha}}. \tag{7}$$

Now, since $b < x_{k+1}^{a_{k+1}+1}$, if we replace x_{k+1} by the value we have just obtained, we have

$$b < \left(b^{\frac{1}{\alpha}}\right)^{a_{k+1}+1} = b^{\frac{a_{k+1}+1}{\alpha}},$$

which implies that

$$1 < \frac{a_{k+1}+1}{\alpha} \Leftrightarrow \alpha < a_{k+1}+1.$$

Finally, since $\alpha > a_k + 1$, we conclude that $a_k < a_{k+1}$.

If in the former reasoning we set $k = n-1$, we can find out what happens with the last two terms when $x_{n+1} = 1$. In that case, we have $x_n^{a_n} = b$ and from (7), which tells us that $x_n = b^{1/\alpha}$, we can say $(b^{1/\alpha})^{a_n} = x^n a_n = b \Rightarrow a_n = \alpha$. On the other hand, since from the definition of α, (6), we have $\alpha > a_{n-1}+1$, we can conclude that $a_{n-1} < a_n - 1$.

Thus, we have proved that the expression in the exponent of b given by (4) is a true Pierce expansion: a terminating one in the case $x_{n+1} = 1$ or a nonterminating one in the case in which, for all $n \in \mathbb{N}$, $x_{n+1} > 1$. In the latter instance, we have to prove that, for $n \to \infty$,

$$b^{\left(\frac{1}{a_1}-\frac{1}{a_1a_2}+\cdots+\frac{(-1)^{n-1}}{a_1a_2\ldots a_n}\right)}\cdot x_{n+1}^{\frac{(-1)^n}{a_1\ldots a_n}} \to b^{\langle a_1,\ldots,a_n,\ldots\rangle}$$

or, equivalently

$$\lim_{n\to\infty} x_{n+1}^{\frac{(-1)^n}{a_1\ldots a_n}} = 1. \tag{8}$$

It is clear that the sequence $\{x_{n+1}^{(-1)^n/(a_1\ldots a_n)}\}_n$ can be split into two subsequences,

$$x_3^{\frac{1}{a_1a_2}},\ x_5^{\frac{1}{a_1a_2a_3a_4}},\ldots \quad\text{and}\quad x_2^{-\frac{1}{a_1}},\ x_4^{-\frac{1}{a_1a_2a_3}},\ldots, \tag{9}$$

and since $\forall n,\ 1 < x_n < b$, and $a_1a_2\ldots a_n \to \infty$, both subsequences in (9) have limit 1, thus proving (8) along with

$$x = b^{\langle a_1,\ldots,a_n,\ldots\rangle} \Leftrightarrow \log_b x = \langle a_1,\ldots,a_n,\ldots\rangle.\ \square$$

3. PRACTICAL USE OF THE ALGORITHM

The present algorithm is purely theoretical and of little practical interest. The difficulties of carrying out the calculations involved are quire important due to the size of the integers appearing in them. In that sense, Shanks' algorithm is much easier to use, thanks mainly to the actual distribution of partial quotients in a continued fraction in which a given integer k occurs with the approximate probability,

$$\log_2 \frac{(1+k)^2}{(1+k)^2-1}$$

(see [4], pp. 351-52), thus making small integers much more abundant and, consequently, calculations much simpler. Let us consider the following example.

Example: Pierce expansion of $\log_{10} 2 = 0.30102999\ldots$. We have:

$$x_1 = 2; \quad a_1 = 3; \quad x_2 = \frac{5}{4}; \quad a_2 = 10;$$

$$x_3 = \frac{2097152}{1953125}; \quad a_3 = 32; \quad x_4 = \frac{10\cdot 1953125^{32}}{2097152^{32}}; \quad a_4 = 89;$$

and

$$\langle 3,10,32\rangle = \frac{289}{960} = 0.30104\ldots; \quad \langle 3,10,32,89\rangle = \frac{643}{2136} = 0.30102996\ldots .$$

Using Shanks' algorithm, we would obtain

$$\log_{10} 2 = [0; 3,3,9,2,2,\ldots] = \cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}}.$$

In this case, the sixth convergent is $146/485 = 0.30103092783\ldots$. As Olds mentions (see [5], p. 87), each convergent approximates log 2 to one more correct decimal place than the previous one.

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