THE FACTORIZATION OF $x^5 \pm p^2x - k$ AND FIBONACCI NUMBERS

Piero Filipponi

Fondazione Ugo Bordoni, Via B. Castiglione 59, I-00142 Rome, Italy
e-mail: filippo@fub.it

Michele Elia

Dip. di Elettronica, Politecnico di Torino, C.so Duca degli Abruzzi 24, I-10129 Torino, Italy
e-mail: elia@polito.it

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1. AIM OF THE PAPER

Here we extend a result established by Rabinowitz [6] by considering the fifth-degree polynomials of the so-called Bring-Jerrard form $q(x, h, k) := x^5 \pm h^2x - k$, where h is either 1 or a prime, and k is an integer. More precisely, the principal aim of the paper is to find necessary and sufficient conditions on k for $q(x, h, k)$ to factor over $\mathbb{Z}$.

Since $q(x, h, k)$ factors trivially as

$$x^5 \pm h^2x - (m^5 \pm h^2m) = (x-m)(x^4 + mx^3 + m^2x^2 + m^3x + m^4 \pm h^2) \tag{1.1}$$

if $k = m^5 \pm h^2m$ $(m \in \mathbb{Z})$, we are concerned with the factorizations of $q(x, h, k)$ that have the form

$$q(x, h, k) = (x^2 + ax + b)(x^3 - ax^2 + cx + d) \quad (a, b, c, d \in \mathbb{Z}). \tag{1.2}$$

The case $h = 1$ has been solved brilliantly by Rabinowitz in [6] (see also [3] and [9]), where he shows that $q(x, 1, k)$ has the factorization (1.2) iff k assumes some special values depending on square Fibonacci numbers. In the more general situation (h a prime), certain properties of the Fibonacci (and generalized Fibonacci) numbers play a crucial role as well.

After observing that changing the sign of k implies nothing but the sign change of a and d in (1.2), we can assume that $k \geq 1$ without loss of generality. Consequently, we shall confine ourselves to studying the factorization (1.2) of the polynomials

$$\begin{cases} r(x, p, k) = x^5 - p^2x - k, \\ s(x, p, k) = x^5 + p^2x - k, \end{cases} \quad (k \geq 1, p \text{ a prime}). \tag{1.3}$$

As will be shown in the sequel, it is necessary to distinguish three cases depending on whether the prime p is either 5, or has the form $5j \pm 2$, or the form $5j \pm 1$. Our approach to this problem will follow [3] and Rabinowitz' argumentation but, to render the paper self-contained, the proofs will be given in full detail. For the sake of completeness, the most significant factorizations will be explicitly shown. A brief discussion on the factorization of $r(x, p, k)$ for certain special primes p concludes our study.

It must be noted that some questions remain unsettled that are related to well-known open problems in number theory. Namely, they concern the existence of infinitely many prime Fibonacci numbers, the occurrence of perfect squares in terms of Fibonacci-like sequences, and the solution of a special Pell equation.

A preliminary version of this paper has been presented by the first author at the XIV Österreichischer Mathematikerkongress [4].

2. PRELIMINARY RESULTS

Given the factorization (1.2), by equating the coefficients of like powers of x we obtain the system

$$\begin{cases} b+c-a^2=0, \\ a(b-c)-d=0, \\ ad+bc=\pm p^2, \\ bd=-k, \end{cases} \tag{2.1}$$

whence, by using the first two equations to eliminate a and d, we obtain the two equations

$$\begin{cases} b^2+bc-c^2=\pm p^2, \\ b^2(b-c)^2(b+c)=k^2. \end{cases} \tag{2.2}$$

Equations (2.2) show that the couple (b, c) must be chosen among the couples that represent $\pm p^2$ by means of the quadratic form $Q(b,c)=b^2+bc-c^2$, subject to the condition that $b+c$ is a perfect square. Hence, finding the solutions of the quadratic equation $Q(b,c)=\pm p^2$ is clearly a necessary step to solve our problem. From Gauss's general theory of the quadratic forms, it is known (e.g., see [5]) that there is a finite number of classes of solutions. Each class consists of an infinitude of solutions which are referred to as *associated solutions*, and is characterized by a single solution called the *fundamental solution*. The classification of the solutions of $Q(b, c) = M$ is given by Dodd in [2]. It depends on the peculiar properties of $\mathbb{Z}(\alpha)$, the ring of integers in the quadratic field $\mathbb{Q}(\alpha)$ which is the extension of the rational field $\mathbb{Q}$ by means of the golden section $\alpha=(1+\sqrt{5})/2$. Recall that $\mathbb{Z}(\alpha)$ is a unique factorization domain.

Every solution (x_n, y_n) of $Q(b, c) = M$, associated to a given fundamental solution (x_0, y_0), is obtained as

$$x_n+\alpha y_n=\alpha^{2n}(x_0+\alpha y_0). \tag{2.3}$$

Equivalently, we can say that both the sequences $\{x_n\}$ and $\{y_n\}$ are generalized Fibonacci sequences obeying the second-order recurrence

$$G_n=G_{n-1}+G_{n-2}, \tag{2.4}$$

with suitable initial conditions G_0 and G_1. The number of classes of solutions is obtained as a consequence of Theorem 3.12 and Corollary 3.13 of [2] that we quote as a single theorem for ease of reference.

Theorem 1 (Dodd): The quadratic equation $x^2+xy-y^2=M$ is solvable in $\mathbb{Z}$ iff

$$M=\pm 5^t p_1^{2f_1}\cdots p_s^{2f_s} q_1^{g_1}\cdots q_r^{g_r} \quad (t,\, f_i,\, g_i \text{ nonnegative integers}),$$

where $p_i=5j\pm 2$ $(1\le i\le s)$ and $q_i=5j\pm 1$ $(1\le i\le r)$ are primes. The number of fundamental solutions is given by the product $(g_1+1)(g_2+1)\cdots(g_r+1)$.

Consequently, for our special case $M=\pm p^2$ [see (2.2)], we can summarize the above results as follows.

(i) If $p=5$ or $5j\pm 2$, then there is a unique fundamental solution

$$(x_0, y_0)=\begin{cases} (p, 0) & \text{if } M=p^2, \\ (0, p) & \text{if } M=-p^2. \end{cases} \tag{2.5}$$

(ii) If $p = 5j \pm 1$, then there are three fundamental solutions, one of which is given by (2.5). The additional solutions $(x_0^{(1)}, y_0^{(1)})$ and $(x_0^{(2)}, y_0^{(2)})$ can be derived from a solution (u_0, v_0) of the Pell equation $u^2 - 5v^2 = p$. Namely, we have

$$\begin{cases} x_0^{(1)} = u_0^2 - 2u_0v_0 + 5v_0^2, \\ y_0^{(1)} = 4u_0v_0; \end{cases} \quad \begin{cases} x_0^{(2)} = u_0^2 + 2u_0v_0 + 5v_0^2, \\ y_0^{(2)} = -4u_0v_0, \end{cases} \text{if } M = p^2; \tag{2.6}$$

$$\begin{cases} x_0^{(1)} = 4u_0v_0, \\ y_0^{(1)} = u_0^2 + 2u_0v_0 + 5v_0^2, \end{cases} \quad \begin{cases} x_0^{(2)} = -4u_0v_0, \\ y_0^{(2)} = u_0^2 - 2u_0v_0 + 5v_0^2, \end{cases} \text{if } M = -p^2. \tag{2.7}$$

Apparently, there is no direct technique for solving the Pell equation $u^2 - 5v^2 = p$; the best known method (see [5], p. 206) is to check every u lying within the interval $[\sqrt{p+5}, \sqrt{5p}]$.

3. THE FACTORIZATION OF $r(x, p, k)$ WHEN $p = 5j \pm 2$

We state the following theorem.

Theorem 2: If $p = 5j \pm 2$, then the polynomial $r(x, p, k)$ given by (1.3) factors as (1.2) iff

$$p = \begin{cases} 2, \text{ and } k = 96 \text{ or } 11424, \\ 3, \text{ and } k = 27 \text{ or } 2808. \end{cases} \tag{3.1}$$

Proof: The system (2.2) becomes

$$\begin{cases} bc + b^2 - c^2 = -p^2, \\ b^2(b-c)^2(b+c) = k^2. \end{cases} \tag{3.2}$$

Since the couple $(0, p)$ is the fundamental solution [see (2.5)] of $Q(b,c) = -p^2$, from (2.3) we know that all the solutions are given by

$$(b, c) = \pm(pF_{2n}, pF_{2n+1}) \quad (n \in \mathbb{Z}), \tag{3.3}$$

where F_n is the n^{th} Fibonacci number. We recall that $F_{-n} = (-1)^{n+1}F_n$.

From (3.3) and the second equation of (3.2), we see that

$$k^2 = p^4 F_{2n}^2 F_{2n-1}^2 (\pm pF_{2n+2}), \tag{3.4}$$

where the minus sign in the last factor must occur iff $n < -1$. From (3.4), it is plain that, for k to be an integer, pF_{2n+2} must be a perfect square. In turn, this implies that we must have

$$F_{2n+2} = py^2. \tag{3.5}$$

For $p = 2$, Theorem 4 of [1] tells us that the only nonzero solution to (3.5) is $F_6 = 2 \cdot 2^2$ [i.e., $y = 2$ and $n = 2$ in (3.5)]. Consequently, letting $n = p = 2$ in (3.4) yields

$$k = \sqrt{16F_4^2F_3^2(2F_6)} = 96. \tag{3.6}$$

Further, letting $n = -4$ in (3.4) (so that the last factor therein becomes $-2F_{-6}$) yields

$$k = \sqrt{16F_{-8}^2F_{-9}^2(-2F_{-6})} = 11424. \tag{3.7}$$

For $p \geq 3$, Theorem 1 of [8] tells us that the unique solution to (3.5) is $F_4 = 3 \cdot 1^2$ [i.e., $p = 3$, $y = 1$, and $n = 1$ in (3.5)]. Hence, letting $p = 3$ and $n = 1$ in (3.4) yields

$$k = \sqrt{81F_2^2F_1^2(3F_4)} = 27. \tag{3.8}$$

Further, letting $n = -3$ in (3.4) (so that the last factor therein becomes $-3F_{-4}$) yields

$$k = \sqrt{81F_{-6}^2F_{-7}^2(-3F_{-4})} = 2808. \text{ Q.E.D.} \tag{3.9}$$

By using (1.2), (2.1), and (3.3), the factorizations of $r(x, 2, k)$ and $r(x, 3, k)$ for the above values of k are readily obtained. Namely, we get

$$x^5 - 4x - 96 = (x^2 + 4x + 6)(x^3 - 4x^2 + 10x - 16), \tag{3.10}$$

$$x^5 - 4x - 11424 = (x^2 - 4x + 42)(x^3 + 4x^2 - 26x - 272), \tag{3.11}$$

$$x^5 - 9x - 27 = (x^2 + 3x + 3)(x^3 - 3x^2 + 6x - 9), \tag{3.12}$$

$$x^5 - 9x - 2808 = (x^2 - 3x + 24)(x^3 + 3x^2 - 15x - 117). \tag{3.13}$$

4. THE FACTORIZATION OF $s(x, p, k)$ WHEN $p = 5j \pm 2$

We state the following theorem.

Theorem 3: If $p = 5j \pm 2$, then the polynomial $s(x, p, k)$ given by (1.3) factors as (1.2) iff

$$p = F_{2n+1} \text{ is a prime Fibonacci number, and } k = \begin{cases} p^3F_{2n-1}F_{2n-2}, \\ p^3F_{2n+3}F_{2n+4}. \end{cases} \tag{4.1}$$

Remark 1: For F_{2n+1} to be a prime, $2n+1$ must necessarily be a prime. The question of whether there exist infinitely many prime Fibonacci numbers is still unsolved ([7], p. 226).

Proof: The system (2.2) becomes

$$\begin{cases} bc + b^2 - c^2 = p^2, \\ b^2(b-c)^2(b+c) = k^2. \end{cases} \tag{4.2}$$

Since the couple $(p, 0)$ is the fundamental solution [see (2.5)] of $Q(b, c) = p^2$, from (2.3) we know that all the solutions are given by

$$(b, c) = \pm(pF_{2n-1}, pF_{2n}) \quad (n \in \mathbb{Z}). \tag{4.3}$$

From (4.3) and the second equation of (4.2), we see that

$$k^2 = p^4F_{2n-1}^2F_{2n-2}^2(pF_{2n+1}), \tag{4.4}$$

where one can observe the absence of the minus sign in the last factor which is due to the fact that the odd-subscripted Fibonacci numbers are always positive. From (4.4), it is plain that, for k to be an integer, pF_{2n+1} must be a perfect square. In turn, this implies that we must have

$$F_{2n+1} = py^2. \tag{4.5}$$

Theorem 2 of [8] ensures us that, if $p = 5j \pm 2$, then all the solutions to (4.5) are given (trivially) by

$$F_{2n+1} = p \cdot 1^2. \tag{4.6}$$

From (4.6), expression (4.4) becomes

$$k^2 = p^6F^2_{2n-1}F^2_{2n-2}, \tag{4.7}$$

whence one immediately gets the first equality of (4.1). Since $F_{-(2n+1)} = F_{2n+1}$, we can replace n by $-(n+1)$ in (4.4), thus getting [see (4.6)]

$$k^2 = p^4F^2_{-2n-3}F^2_{-2n-4}(pF_{2n+1}) = p^6F^2_{2n+3}F^2_{2n+4}, \tag{4.8}$$

whence the second equality of (4.1) is readily obtained. Q.E.D.

In the first and second cases of (4.1), the factorizations (1.2) of $s(x, p, k)$ have the sets of coefficients

$$\begin{cases} a = p = F_{2n+1}, \\ b = pF_{2n-1}, \\ c = pF_{2n}, \\ d = -p^2F_{2n-2}, \end{cases} \quad \text{and} \quad \begin{cases} a = -p, \\ b = pF_{2n+3}, \\ c = -pF_{2n+2}, \\ d = -p^2F_{2n+4}, \end{cases} \tag{4.9}$$

respectively. As a numerical example, the factorizations of $s(x, 13, k)$ [$n = 3$ in (4.1)] are shown below. Namely, we have [cf. (4.9)]:

$$x^5 + 169x - 32955 = (x^2 + 13x + 65)(x^3 - 13x^2 + 104x - 507), \tag{4.10}$$

$$x^5 + 169x - 4108390 = (x^2 - 13x + 442)(x^3 + 13x^2 - 273x - 9295). \tag{4.11}$$

Remark 2: The case $p = 5$ is exceptional because 5 occurs in the definition of the quadratic extension ring $\mathbb{Z}(\alpha)$, but according to Theorem 1, it can be treated as the primes of the form $5j \pm 2$. Equation $Q(b, c) = 5^2$ has only one fundamental solution, and, according to the above discussion, we get the only possible factorizations:

$$x^5 + 25x - 250 = (x^2 + 5x + 10)(x^3 - 5x^2 + 15x - 25), \tag{4.12}$$

$$x^5 + 25x - 34125 = (x^2 - 5x + 65)(x^3 + 5x^2 - 40x - 525). \tag{4.13}$$

5. THE FACTORIZATION OF $r(x, p, k)$ WHEN $p = 5j \pm 1$

From Theorem 1, we know that, if $p = 5j \pm 1$, then the equation $Q(b, c) = -p^2$ has the three fundamental solutions

$$(b, c) = \begin{cases} \pm(pF_{2n}, pF_{2n+1}), \\ \pm(A_{2n}, A_{2n+1}), \\ \pm(B_{2n}, B_{2n+1}), \end{cases} \quad n \in \mathbb{Z}, \tag{5.1}$$

where the generalized Fibonacci sequences $\{A_n\}$ and $\{B_n\}$ obey the recurrence (2.4) with initial conditions [$A_0 = x_0^{(1)}$; $A_1 = y_0^{(1)}$] and [$B_0 = x_0^{(2)}$; $B_1 = y_0^{(2)}$] that can be obtained from (2.7).

We now state the following theorem.

Theorem 4: If $p = 5j \pm 1$, then the polynomial $r(x, p, k)$ given by (1.3) factors as (1.2) iff A_{2n} and/or B_{2n} are perfect squares for some n.

Proof: On the basis of the previously used arguments [see (3.2)-(3.4)], from (5.1) it is clear that we must have

$$k^2 = \begin{cases} p^4F_{2n}^2F_{2n-1}^2(\pm pF_{2n+2}), \\ A_{2n}^2A_{2n-1}^2(\pm A_{2n+2}), & n \in \mathbb{Z}. \\ B_{2n}^2B_{2n-1}^2(\pm B_{2n+2}), \end{cases} \tag{5.2}$$

The equation $\pm F_{2n+2} = py^2$ [cf. (3.5)] has no solutions by virtue of Theorem 1 of [8]. Therefore, the only possibilities for k to be an integer are that A_{2n+2} and/or B_{2n+2} are perfect squares for some n. Q.E.D.

As a numerical example, let us find values of k for which $r(x, 11, k)$ factors as (1.2). If $p = 11$, then $(u_0, v_0) = (4, 1)$ is a solution of the Pell equation at point (ii) of Section 2, so that expressions (2.7) give the initial conditions $[A_0 = 16; A_1 = 29]$ and $[B_0 = -16; B_1 = 13]$. From (5.2) and the argument in the proof of Theorem 4 (namely, Theorem 1 of [8]), we have

$$k^2 = \begin{cases} A_{2n}^2A_{2n-1}^2(\pm A_{2n+2}) & \text{(the minus sign when } n \le -3), \\ B_{2n}^2B_{2n-1}^2(\pm B_{2n+2}) & \text{(the minus sign when } n \le 0). \end{cases} \tag{5.3}$$

For $n = -1$, we have that $A_{2n+2} = A_0 = 16$ is a perfect square. Letting $n = -1$ in the first equation of (5.3) yields

$$k = A_{-2}A_{-3}\sqrt{A_0} = 3 \cdot 10 \cdot 4 = 120. \tag{5.4}$$

For the same value of n, we see that $B_{2n+2} = B_0 = -16$. Letting $n = -1$ in the second equation of (5.3) and choosing the proper signs yields

$$k = -B_{-2}B_{-3}\sqrt{-B_0} = 45 \cdot 74 \cdot 4 = 13320. \tag{5.5}$$

Remark 3: The occurrence of further even-subscripted terms of $\{A_n\}$ and/or $\{B_n\}$ that are perfect squares would allow us to find further values of k for which $r(x, 11, k)$ factors as (1.2).

The factorizations of $r(x, 11, k)$ for the values of k given by (5.4) and (5.5) are

$$x^5 - 121x - 120 = (x^2 + 4x + 3)(x^3 - 4x^2 + 13x - 40) \tag{5.6}$$

and

$$x^5 - 121x - 13320 = (x^2 - 4x + 45)(x^3 + 4x^2 - 29x - 296), \tag{5.7}$$

respectively.

6. THE FACTORIZATION OF $s(x, p, k)$ WHEN $p = 5j \pm 1$

From Theorem 1, we know that, if $p = 5j \pm 1$, then the equation $Q(b, c) = p^2$ has the three fundamental solutions

$$(b, c) = \begin{cases} \pm(pF_{2n-1}, pF_{2n}), \\ \pm(A_{2n}, A_{2n+1}), & n \in \mathbb{Z}, \\ \pm(B_{2n}, B_{2n+1}), \end{cases} \tag{6.1}$$

where the initial conditions for $\{A_n\}$ and $\{B_n\}$ can be obtained from (2.6).

Now, let us state the following theorem.

Theorem 5: If $p = 5j \pm 1$, then the polynomial $s(x, p, k)$ given by (1.3) factors as (1.2) iff either ***(i)*** (4.1) is satisfied (with $p = 5j \pm 1$) or ***(ii)*** A_{2n+1} and/or B_{2n+1} are perfect squares for some n.

N.B. There is a unique exception to point (i). Namely, $s(x, 3001, k)$ factors as (1.2) for $k = 68586998444168435635$ or $k = 843564315724789391 4990$.

After observing that, on the basis of previously used arguments [see (4.2)-(4.4)], we must have

$$k^2 = \begin{cases} p^4F_{2n-1}^2F_{2n-2}^2(pF_{2n+1}), \\ A_{2n}^2A_{2n-1}^2(\pm A_{2n+2}), \qquad n \in \mathbb{Z}, \\ B_{2n}^2B_{2n-1}^2(\pm B_{2n+2}), \end{cases} \tag{6.2}$$

it is clear that the proofs of points (i) and (ii) are similar to those of Theorems 3 and 4, respectively. Therefore, we shall confine ourselves to proving the exception to point (i) mentioned in the N.B. above.

As an example of application of the last two equations of (6.2), we invite the reader to prove that $s(x, 19, k)$ factors as (1.2) for $k = 765$ or 26390 or 37704147.

Hint: After assuming that $(u_0, v_0) = (12, 5)$ is a solution to $u^2 - 5v^2 = 19$, use (2.6) to find $[A_0 = 149, A_1 = 240]$ and $[B_0 = 389, B_1 = -240]$, and observe that $A_{-4} = B_6 = 25$ and $B_{20} = 2809$ are perfect squares.

Proof of the Exception to Point (i): Theorem 2 of [8] tells us that the unique exception to (4.6) occurs when $n = 12$, $p = 3001$, and $y = 5$ in (4.5). If we let these values of n and p in the first equation of (6.2), then we get $k^2 = p^4F_{23}^2F_{22}^2(pF_{25})$ $(p = 3001)$, whence

$$k = 3001^2F_{23}F_{22}\sqrt{3001F_{25}} = 68586998444168435635. \tag{6.3}$$

Further, letting $n = -13$ and $p = 3001$ in the same equation yields $k^2 = p^4F_{-27}^2F_{-28}^2(pF_{-25})$ $(p = 3001)$, whence

$$k = 3001^2F_{27}F_{28}\sqrt{3001F_{25}} = 8435643157247893914990. \text{ Q.E.D.} \tag{6.4}$$

The factorizations of $s(x, 3001, k)$ for the values of k given by (6.3) and (6.4) are:

$$\begin{aligned} &x^5 + 9006001x - 68586998444168435635 \\ &= (x^2 + 15005x + 85999657)(x^3 - 15005x^2 + 139150368x - 797526418555), \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} &x^5 + 9006001x - 8435643157247893914990 \\ &= (x^2 - 15005x + 589450418)(x^3 + 15005x^2 - 364300393x - 14311030919055), \end{aligned} \tag{6.6}$$

respectively.

7. CONCLUSIONS

First, we wish to point out that the technique used in Sections 3-6 allows us to obtain the factorization of fifth-degree polynomials that are similar to those considered in this paper. In every case, Fibonacci and Fibonacci-like sequences play a fundamental role, and suggest the existence of an even deeper connection between these sequences and the factorization of fifth-degree polynomials. For example, it is not hard to prove that if

$$k = \begin{cases} n^{5m}F_4F_5, \\ 12n^{5m}F_9F_{10}, \\ 12n^{5m}F_{14}F_{15}, \end{cases} \tag{7.1}$$

then the polynomials $x^5 - n^{4m}x - k$ $(n, m \in \mathbb{N})$ factor as (1.2) (for $n = 1$, cf. (3.8) of [3]). The proof of (7.1) is based on the well-known fact [1] that F_{2j} is a perfect square iff $j = 0, 1$, or 6. Further, the interested reader might enjoy using the above technique for proving that, if

$$k = F_j^3 F_{j\pm 2} F_{j\pm 3}, \tag{7.2}$$

then the polynomials $x^5 - (-1)^j F_j^2 x - k$ factor as (1.2).

Then, let us conclude our study by considering a special class of primes p such that a couple (k_1, k_2) of values of k for which $r(x, p, k)$ factors as (1.2) can be expressed merely in terms of p. Namely, consider the set of all primes p such that $p + 5 = z^4$ is a fourth power. Since z must be an even integer not divisible by 5, it can be readily proven that p has the form $5j + 1$. It is likely that there exist an infinitude of primes belonging to the above defined set. We found 15 of them within the interval $[2, 10^8]$, the smallest (resp. largest) being 11 (resp. 78074891).

Theorem 6: If $p \geq 251$ is a prime such that

$$p + 5 = z^4 \tag{7.3}$$

is a fourth power, and

$$k_{1,2} = 4(p+5)^{1/4}[p^2 + 44p \pm 10(p+5)^{1/2}(p+10) + 220], \tag{7.4}$$

then both $r(x, p, k_1)$ and $r(x, p, k_2)$ factor as (1.2).

Remark 4: For $p = 11$, see (5.4) and (5.5).

Proof (for $k = k_2$): A solution to the Pell equation at point (ii) of Section 2 is clearly $(u, v) = (z^2, 1)$. Hence, from the first system of (2.7), we have

$$\begin{cases} x_0^{(1)} = A_0 = 4z^2 \quad \text{(a perfect square)}, \\ y_0^{(1)} = A_1 = z^4 + 2z^2 + 5, \end{cases} \tag{7.5}$$

and, from (5.2),

$$k^2 = A_{2n}^2 A_{2n-1}^2 A_{2n+2}. \tag{7.6}$$

Letting $n = -1$ in (7.6) yields

$$k^2 = A_{-2}^2 A_{-3}^2 A_0 = A_{-2}^2 A_{-3}^2 4z^2 \quad \text{[from (7.5)]}. \tag{7.7}$$

On calculation, we get

$$\begin{cases} A_{-2} = -z^4 + 6z^2 - 5, \\ A_{-3} = 2z^4 - 8z^2 + 10. \end{cases} \tag{7.8}$$

From (7.7) and (7.8) above, on choosing the signs properly to ensure the positiveness of k, one gets

$$k = \begin{cases} 2z(z^4 - 6z^2 + 5)(2z^4 - 8z^2 + 10) \quad \text{for } z \geq 4, \\ 2z(-z^4 + 6z^2 - 5)(2z^4 - 8z^2 + 10) = 120 \quad \text{for } z = 2 \text{ (i.e., } p = 11). \end{cases} \tag{7.9}$$

For $z \geq 4$ (i.e., $p \geq 251$), from (7.9) and (7.3), we obtain

$$k = k_2 = 4(p+5)^{1/4}[p^2 + 44p - 10(p+5)^{1/2}(p+10) + 220]$$

as desired. By using the second system of (2.7), the proof for $k = k_1$ can be obtained in a similar way. Q.E.D.

The factorizations (1.2) of $r(x, p, k)$ have the sets of coefficients

$$\begin{cases} a = -2(p+5)^{1/4}, \\ b = p + 6(p+5)^{1/2} + 10, \\ c = -p - 2(p+5)^{1/2} - 10, \\ d = -k/b, \end{cases} \quad \text{and} \quad \begin{cases} a = -2(p+5)^{1/4}, \\ b = -p + 6(p+5)^{1/2} - 10, \\ c = p - 2(p+5)^{1/2} + 10, \\ d = -k/b, \end{cases} \tag{7.10}$$

for $k = k_1$ and k_2, respectively. As a numerical example, the factorizations of $r(x, 1291, k_{1,2})$ are shown below. Namely, we have [cf. (7.10)]

$$x^5 - 1291^2 x - 52609560 = (x^2 - 12x + 1517)(x^3 + 12x^2 - 1373x - 34680), \tag{7.11}$$

$$x^5 - 1291^2 x - 30128280 = (x^2 - 12x - 1085)(x^3 + 12x^2 - 1229x + 27768). \tag{7.12}$$

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