SOME ANALOGS OF THE IDENTITY $F_n^2 + F_{n+1}^2 = F_{2n+1}$

R. S. Melham

School of Mathematical Sciences, University of Technology, Sydney
PO Box 123, Broadway, NSW 2007 Australia
(Submitted December 1997-Final Revision July 1998)

1. INTRODUCTION

The charming identity

$$\sum_{j=0}^{k}(-1)^{\frac{j(j+3)}{2}}\begin{bmatrix}k\\j\end{bmatrix}F_{n+k-j}^{k+1} = F_1 \dots F_k F_{(k+1)(n+\frac{k}{2})} \tag{1.1}$$

is a special case of identity (5) of Torretto and Fuchs [7]. Here $\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]$ is the Fibonomial coefficient defined for integers $0 \le j \le k$ by

$$\begin{bmatrix}k\\j\end{bmatrix} = \frac{F_k F_{k-1} \dots F_{k-j+1}}{F_1 F_2 \dots F_j}, \quad \begin{bmatrix}k\\0\end{bmatrix} = 1.$$

According to H. W. Gould, generalized binomial coefficients were first suggested by Georges Fontené in 1915, and were rediscovered by Morgan Ward in 1936. These writers simply replaced the natural numbers by an arbitrary sequence $\{A_n\}$ of real or complex numbers. The idea of considering $A_n = F_n$ seems to have originated with Dov Jarden in 1949. For an excellent discussion on these matters, and a comprehensive list of references, see Gould [3].

For $k = 1, 2, 3$, and 4, identity (1.1) becomes, respectively,

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \tag{1.2}$$

$$F_{n+2}^3 + F_{n+1}^3 - F_n^3 = F_{3n+3}, \tag{1.3}$$

$$F_{n+3}^4 + 2F_{n+2}^4 - 2F_{n+1}^4 - F_n^4 = 2F_{4n+6}, \tag{1.4}$$

$$F_{n+4}^5 + 3F_{n+3}^5 - 6F_{n+2}^5 - 3F_{n+1}^5 + F_n^5 = 6F_{5n+10}. \tag{1.5}$$

To make the right sides of (1.3) and (1.5) more compact, we may replace n by $n-1$ and $n-2$, respectively.

In this paper we present analogs of (1.2)–(1.5) for the so-called Tribonacci and Tetranacci sequences, which we define in Sections 3 and 4. We consider more general third- and fourth-order sequences, and identities associated with them, in Section 5. Our method of discovering these identities is outlined in Section 2, and generalizations and proofs are given in Section 6.

2. THE METHOD

To demonstrate our method, we use it to "discover" identities (1.2) and (1.3). To arrive at (1.2), we consider the sequence

$$\{F_n^2 - F_{n+1}F_{n-1}\}_0^\infty = \{-1, 1, -1, 1, -1, \dots\}.$$

This sequence satisfies the recurrence $r_n = -r_{n-1}$, and so we have

$$F_n^2 - F_{n+1}F_{n-1} = -(F_{n-1}^2 - F_nF_{n-2})$$

or

$$F_n^2 + F_{n-1}^2 = F_{n+1}F_{n-1} + F_nF_{n-2}. \tag{2.1}$$

Finally, we observe by trial that the right side of (2.1) is F_{2n-1}, and this yields (1.2).

To obtain (1.3), we consider the sequence

$$\{F_n^3 - F_{n+1}F_nF_{n-1}\}_0^\infty = \{0, 1, -1, 2, -3, 5, -8, \ldots\}.$$

This sequence satisfies the recurrence $r_n = -r_{n-1} + r_{n-2}$, so that

$$F_n^3 - F_{n+1}F_nF_{n-1} = -(F_{n-1}^3 - F_nF_{n-1}F_{n-2}) + (F_{n-2}^3 - F_{n-1}F_{n-2}F_{n-3})$$

or

$$F_n^3 + F_{n-1}^3 - F_{n-2}^3 = F_{n+1}F_nF_{n-1} + F_nF_{n-1}F_{n-2} - F_{n-1}F_{n-2}F_{n-3}. \tag{2.2}$$

Again, after making several substitutions, we see that the right side of (2.2) is F_{3n-3}, and this yields (1.3).

To obtain (1.4), we could consider the sequence generated by $F_n^4 - F_{n+1}F_n^2F_{n-1}$, or perhaps $F_n^4 - F_{n+3}F_nF_{n-1}F_{n-2}$, or many other such expressions. To decide which product to subtract, we consider two things. First, the product must have "degree" four. Second, the sum of the subscripts of the terms which make up the product must be $4n$. To obtain the analogous identities which involve higher powers, we proceed in a similar manner.

3. THE TRIBONACCI SEQUENCE

As a third-order analog of the Fibonacci sequence, Feinberg [2] considered the Tribonacci sequence, defined for all integers by

$$p_n = p_{n-1} + p_{n-2} + p_{n-3}, \quad p_0 = 0, p_1 = 1, p_2 = 1.$$

Proceeding as in Section 2, and with the help of the computer algebra package Mathematica 3.0, we have obtained identities analogous to (1.2)–(1.5) for the Tribonacci sequence. We have found the following:

$$p_{n+3}^2 + p_{n+2}^2 + p_{n+1}^2 - p_n^2 = 2p_{2n} + 3p_{2n+1} + 3p_{2n+2}, \tag{3.1}$$

$$\begin{aligned} &p_{n+7}^3 + 3p_{n+6}^3 + 7p_{n+5}^3 + p_{n+4}^3 - p_{n+3}^3 - 7p_{n+2}^3 - 3p_{n+1}^3 - p_n^3 \\ &= 6758p_{3n} + 10432p_{3n+1} + 12430p_{3n+2}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &p_{n+12}^4 + 4p_{n+11}^4 + 16p_{n+10}^4 - 26p_{n+9}^4 - 5p_{n+8}^4 - 128p_{n+7}^4 \\ &+ 100p_{n+6}^4 + 4p_{n+5}^4 + 43p_{n+4}^4 - 44p_{n+3}^4 + 4p_{n+2}^4 - 2p_{n+1}^4 + p_n^4 \\ &= 27720670104p_{4n} + 42792093864p_{4n+1} + 50986261368p_{4n+2}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &p_{n+18}^5 + 8p_{n+17}^5 + 59p_{n+16}^5 - 126p_{n+15}^5 - 154p_{n+14}^5 - 2758p_{n+13}^5 \\ &+ 2142p_{n+12}^5 + 2394p_{n+11}^5 + 6552p_{n+10}^5 - 7182p_{n+9}^5 - 4284p_{n+8}^5 - 2394p_{n+7}^5 \\ &+ 1386p_{n+6}^5 + 686p_{n+5}^5 + 322p_{n+4}^5 - 98p_{n+3}^5 - 9p_{n+2}^5 - 2p_{n+1}^5 + p_n^5 \\ &= 1252886775213004795584p_{5n} + 1934067549043522783296p_{5n+1} \\ &+ 2304418051432261675008p_{5n+2}. \end{aligned} \tag{3.4}$$

We have found the next identity in this list. The left side has 26 sixth powers, and following the pattern of the previous identities the coefficients are 1, 15, 204, –724, –1946, –58710, 65968, 182480, 921767, –1448495, –2215192, –2814392, 1090180, 2032604, 2528400, –9744, –25313, –238687, –15828, –4372, 9814, 1786, 224, –32, –7, –1. On the right side, the coefficients of p_{6n}, p_{6n+1}, and p_{6n+2} are, respectively,

$$
\begin{aligned}
&32119103347966496693731741070891558 40,\\
&49581906935777165672226969703584998 40,\\
\text{and}\quad &59076241377269597102082587261723481 60.
\end{aligned}
$$

We have been unable to discern a pattern to the coefficients in the identities above. However, on the basis of our results, we predict that the next identity will involve 34 seventh powers. More generally, we conjecture that for $k \geq 2$ such identities for the Tribonacci sequence involve $\frac{1}{2}(k^2+3k-2)$ k^{th} powers.

4. THE TETRANACCI SEQUENCE

As a fourth-order analog of the Fibonacci sequence, Feinberg [2] also considered the Tetranacci sequence, defined for all integers by

$$q_n = q_{n-1}+q_{n-2}+q_{n-3}+q_{n-4}, \quad q_0=0, q_1=1, q_2=1, q_3=2.$$

In the same manner, for the Tetranacci sequence, we have found

$$
\begin{aligned}
&q_{n+6}^2+q_{n+5}^2+2q_{n+4}^2+2q_{n+3}^2-2q_{n+2}^2+q_{n+1}^2-q_n^2\\
&=46q_{2n}+70q_{2n+1}+82q_{2n+2}+88q_{2n+3}
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&q_{n+16}^3+3q_{n+15}^3+8q_{n+14}^3+18q_{n+13}^3-26q_{n+12}^3-35q_{n+11}^3-q_{n+10}^3-56q_{n+9}^3\\
&+36q_{n+8}^3+88q_{n+7}^3-21q_{n+6}^3+21q_{n+5}^3-16q_{n+4}^3-6q_{n+3}^3+2q_{n+2}^3-q_{n+1}^3+q_n^3\\
&=273507715816q_{3n}+415400801120q_{3n+1}+489013523880q_{3n+2}+527203073008q_{3n+3}.
\end{aligned}
\tag{4.2}
$$

The next identity involves 32 fourth powers whose coefficients are 1, 7, 38, 174, –154, –1150, –1368, –7226, –1926, 32582, 22851, 56387, 36788, –34100, –23540, –78932, –56080, 6372, 18724, 50476, 39447, 13621, 2822, –2234, –2290, –910, –280, –10, 34, 14, 5, 1. On the right side, the coefficients of q_{4n}, q_{4n+1}, q_{4n+2}, and q_{4n+3} are, respectively,

$$
\begin{aligned}
&402934710032647317503725654362880,\\
&611973233907708364378185877905536,\\
&720420343019564129073011409939840,\\
\text{and}\quad &776681625661169345246132510366848.
\end{aligned}
$$

We have found that the next identity in this list involves 53 fifth powers. On the basis of our observations, we conjecture that for $k \geq 2$ such identities for the Tetranacci sequence involve $\frac{1}{6}(k^3+6k^2+11k-12)$ k^{th} powers.

5. MORE GENERAL SEQUENCES

Consider now the more general sequence $\{U_n\}$ defined for all integers by

$$U_n = aU_{n-1}+bU_{n-2}+cU_{n-3}, \quad U_0=0, U_1=1, U_2=a, \tag{5.1}$$

where a, b, and c are complex numbers with $c \neq 0$. The sequence $\{U_n\}$ is one of the three *fundamental* sequences (as in Bell [1]) generated by the recurrence in (5.1). We have found that

$$U_{n+3}^2 + bU_{n+2}^2 + acU_{n+1}^2 - c^2U_n^2 = bU_1U_{2n+3} + U_2U_{2n+4}. \tag{5.2}$$

We accomplished this by considering many instances of (a, b, c) and constructing the corresponding identity. This process was tedious, to say the least.

More generally, let $\{R_n\}$ be any sequence generated by the recurrence in (5.1) and with arbitrary initial terms R_0, R_1, R_2. Then, in the same manner, we have found that

$$\begin{aligned} &R_{n+3}^2 + bR_{n+2}^2 + acR_{n+1}^2 - c^2R_n^2 \\ &= ((ac-b^2)R_0 - abR_1 + bR_2)R_{2n+2} + (-abR_0 + (b-a^2)R_1 + aR_2)R_{2n+3} + (bR_0 + aR_1)R_{2n+4}. \end{aligned} \tag{5.3}$$

It is interesting to note that the coefficients on the left side of (5.2) match those on the left side of (5.3). Horadam [5] proved the analog of (5.3) for second-order sequences very elegantly with the use of generating functions, but we have been unable to adapt his method to prove (5.3). However, we have discovered another method of proof which we demonstrate in the next section.

As the fourth-order analog of $\{U_n\}$, we define the sequence $\{V_n\}$ by

$$V_n = aV_{n-1} + bV_{n-2} + cV_{n-3} + dV_{n-4}, \quad V_0 = 0, V_1 = 1, V_2 = a, V_3 = a^2 + b. \tag{5.4}$$

We have found that

$$\begin{aligned} &V_{n+6}^2 + bV_{n+5}^2 + (ac+d)V_{n+4}^2 + (a^2d - c^2 + 2bd)V_{n+3}^2 - (d^2 + acd)V_{n+2}^2 + bd^2V_{n+1}^2 - d^3V_n^2 \\ &= (a^2d - c^2 + 2bd)V_1V_{2n+5} + (ac+d)V_2V_{2n+6} + bV_3V_{2n+7} + V_4V_{2n+8}. \end{aligned} \tag{5.5}$$

In (5.5), it is interesting to compare the coefficients of $V_{n+3}^2, V_{n+4}^2, V_{n+5}^2$, and V_{n+6}^2 with those of $V_{2n+5}, V_{2n+6}, V_{2n+7}$, and V_{2n+8}, respectively. Similar comparisons should be made in (5.2), and also in the known identity

$$u_{n+1}^2 + bu_n^2 = u_{2n+1} = u_1u_{2n+1}. \tag{5.6}$$

Here $\{u_n\}$ is the second-order sequence defined by $u_n = au_{n-1} + bu_{n-2}$, $u_0 = 0, u_1 = 1$.

Our attempt to construct identities similar to those in this section for sequences of order five has proved fruitless. The polynomial coefficients became unwieldy, as can be appreciated when we compare (5.2) with (5.5). The same can be said for higher powers. However, our work with specific examples suggests that identities analogous to those that we have constructed in this paper exist for all sequences, and for all powers. We have looked only at sequences generated by linear recurrence relations with constant coefficients.

We mention that further experimentation with specific examples suggests that, for linear recurrences of order m, identities analogous to (1.2) contain $\frac{1}{2}(m^2 - m + 2)$ squares, and identities analogous to (1.3) contain $\frac{1}{6}(m^3 + 3m^2 - 4m + 6)$ cubes.

6. GENERALIZATIONS AND PROOFS

At the beginning of Section 2 we started with the identity $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$. Instead, suppose we consider the more general identity

$$F_{n+a}F_{n+b} - F_nF_{n+a+b} = (-1)^n F_aF_b. \tag{6.1}$$

Then, considered as a function of n, the sequence $\{F_{n+a}F_{n+b} - F_nF_{n+a+b}\}$ satisfies the recurrence $r_n = -r_{n-1}$. Hence,

$$F_{n+1+a}F_{n+1+b} - F_{n+1}F_{n+1+a+b} = -(F_{n+a}F_{n+b} - F_nF_{n+a+b})$$

or

$$F_{n+1+a}F_{n+1+b} + F_{n+a}F_{n+b} = F_{n+1}F_{n+1+a+b} + F_nF_{n+a+b}. \tag{6.2}$$

With m in place of $n+a$, and n in place of $n+b$, the left side of (6.2) becomes $F_{m+1}F_{n+1} + F_mF_n$. But, by I_{26} in [4], we know that

$$F_{m+1}F_{n+1} + F_mF_n = F_{m+n+1}, \tag{6.3}$$

which generalizes (1.2).

This suggests that to generalize (1.3) we might try

$$F_{k+2}F_{m+2}F_{n+2} + F_{k+1}F_{m+1}F_{n+1} - F_kF_mF_n = F_{k+m+n+3}, \tag{6.4}$$

which is indeed the case. In fact, this mode of generalization extends to (1.1), where the corresponding generalization is a special case of identity (5) of Torretto and Fuchs [7].

Based on numerical evidence, the method of generalization we have just described seems to carry over to all the identities in Sections 3-5. For example, we now prove that

$$p_{m+3}p_{n+3} + p_{m+2}p_{n+2} + p_{m+1}p_{n+1} - p_mp_n = 2p_{m+n} + 3p_{m+n+1} + 3p_{m+n+2}, \tag{6.5}$$

which generalizes (3.1).

Proof of (6.5): Fix m. Each of the sequences $\{p_{n+k}\}$, where $k \in \mathbf{Z}$ is fixed, satisfies the recurrence for the Tribonacci numbers. Hence, by linearity, the sequences

$$\{p_{m+3}p_{n+3} + p_{m+2}p_{n+2} + p_{m+1}p_{n+1} - p_mp_n\} \text{ and } \{2p_{m+n} + 3p_{m+n+1} + 3p_{m+n+2}\} \tag{6.6}$$

also satisfy this recurrence. So, to prove that these sequences are identical, it suffices to prove that they have the same initial terms. That is, it suffices to show that

$$\begin{cases} p_{m+3}p_3 + p_{m+2}p_2 + p_{m+1}p_1 - p_mp_0 = 2p_m + 3p_{m+1} + 3p_{m+2}, \\ p_{m+3}p_4 + p_{m+2}p_3 + p_{m+1}p_2 - p_mp_1 = 2p_{m+1} + 3p_{m+2} + 3p_{m+3}, \\ p_{m+3}p_5 + p_{m+2}p_4 + p_{m+1}p_3 - p_mp_2 = 2p_{m+2} + 3p_{m+3} + 3p_{m+4}. \end{cases}$$

We prove only the last of these, since the proofs of the others are similar. Using the recurrence satisfied by the Tribonacci numbers, we see that $p_{m+3} = p_{m+2} + p_{m+1} + p_m$ and $p_{m+4} = 2p_{m+2} + 2p_{m+1} + p_m$. Also, since $p_2 = 1$, $p_3 = 2$, $p_4 = 4$, and $p_5 = 7$, we substitute and observe that both sides reduce to $11p_{m+2} + 9p_{m+1} + 6p_m$. Since m is arbitrary, this proves (6.5) and hence also (3.1). □

This method of proof applies also to identities (4.1), (5.2), (5.3), and (5.5), since they involve squares. As shown above, we proceed by proving the more general identities obtained by introducing the parameter m. The proof of the generalized version of (5.3), for example, is not much more complicated than the proof demonstrated above. With m fixed, we need to prove

$$\begin{aligned} R_{m+3}R_{n+3} + bR_{m+2}R_{n+2} + acR_{m+1}R_{n+1} - c^2R_mR_n \\ = AR_{m+n+2} + BR_{m+n+3} + CR_{m+n+4}, \end{aligned} \tag{6.7}$$

where A, B, and C are as in (5.3). As in the proof of (6.5), our task is to show that (6.7) holds for $n = 0, 1$, and 2. Thus, for $n = 2$, we need to show

$$R_5 R_{m+3} + bR_4 R_{m+2} + acR_3 R_{m+1} - c^2 R_2 R_m = AR_{m+4} + BR_{m+5} + CR_{m+6}. \tag{6.8}$$

Using the recurrence in (5.1), we express R_3, R_4, and R_5 in terms of R_0, R_1, and R_2. Likewise, we express R_{m+j} for $3 \le j \le 6$ in terms of R_m, R_{m+1}, and R_{m+2}. Finally, making these substitutions and using a suitable computer algebra package (in our case Mathematica 3.0), it is straightforward to verify the validity of (6.8). The verifications for $n = 0$ and 1 are treated similarly.

Now to the identities which involve higher powers. We tried to prove (3.2) by first proving

$$\sum_{i=0}^{7} a_i p_{k+i} p_{m+i} p_{n+i} = \sum_{i=0}^{2} b_i p_{k+m+n+i}, \tag{6.9}$$

where the a_i and b_i are given in (3.2). Our attempts failed because of the presence of an extra parameter. However, we found that we could prove the following "intermediate" identity:

$$\sum_{i=0}^{7} a_i p_{m+i} p_{n+i}^2 = \sum_{i=0}^{2} b_i p_{m+2n+i}. \tag{6.10}$$

Our proof, which is similar to the proofs demonstrated previously, requires the following lemma which is contained in [6].

Lemma: Let $\{w_n\}$ be a sequence of complex numbers defined by

$$w_n = \sum_{i=1}^{k} c_i w_{n-i}, \tag{6.11}$$

where $c_1, \ldots, c_k$ and $w_0, \ldots, w_{k-1}$ are given complex numbers with $c_k \ne 0$. Let $h \ge 1$ be an integer. Then $\{w_n^h\}$ is generated by a linear recurrence of order $\binom{h+k-1}{h}$.

Using the lemma with $h = 2$ and $k = 3$, we see that $\{p_n^2\}$ satisfies a linear recurrence of order 6, and, by solving a system of linear equations, we find that this recurrence is

$$r_n = 2r_{n-1} + 3r_{n-2} + 6r_{n-3} - r_{n-4} - r_{n-6}. \tag{6.12}$$

Furthermore, $\{p_{2n}\}$ satisfies the recurrence

$$r_n = 3r_{n-1} + r_{n-2} + r_{n-3}, \tag{6.13}$$

and, since the auxiliary polynomial of (6.13) divides the auxiliary polynomial of (6.12), the sequence $\{p_{2n}\}$ is also generated by (6.12). To complete the proof, we proceed as before. That is, we fix m and verify the validity of (6.10) for six consecutive values of n.

By using this approach, we have also succeeded in proving (3.3), (3.4), and (4.2) by first proving the more general identities obtained by the introduction of the parameter m. From the lemma, the number of verifications required to prove each of these identities is 10, 15, and 10, respectively.

While we acknowledge that this method of proof is tedious for identities that involve higher powers, given the nature of these identities, it seems unreasonable to expect anything else.

ACKNOWLEDGMENT

We are indebted to an anonymous referee whose suggestions led to the generalizations in Section 6, which in turn inspired us to discover our method of proof.

REFERENCES

1. E. T. Bell. "Notes on Recurring Series of the Third Order." *The Tôhoku Mathematical J.* **24.1, 2** (1924):168-84.
2. M. Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* **1.3** (1963):70-74.
3. H. W. Gould. "The Bracket Function and Fontené-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients." *The Fibonacci Quarterly* **7.1** (1969):23-40, 55.
4. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969; rpt. The Fibonacci Association, 1979.
5. A. F. Horadam. "Generating Functions for Powers of a Certain Generalized Sequence of Numbers." *Duke Math. J.* **32** (1965):437-46.
6. A. G. Shannon. "Explicit Expressions for Powers of Linear Recursive Sequences." *The Fibonacci Quarterly* **12.3** (1974):281-87.
7. R. F. Torretto & J. A. Fuchs. "Generalized Binomial Coefficients." *The Fibonacci Quarterly* **2.4** (1964):296-302.

AMS Classification Numbers: 11B37, 11B39

❖❖❖