ON IRRATIONAL VALUED SERIES INVOLVING GENERALIZED FIBONACCI NUMBERS

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(*Submitted November 1997-Final Revision June 1998*)

1. INTRODUCTION

There are many rational termed convergent series in analysis that sum to an irrational number. One well-known example can be found via the Taylor expansion of the exponential function, where in particular the base of the natural logarithm is represented as an infinite sum of the reciprocals of $n!$. The irrationality of e can be deduced directly from this series via an argument of Euler's (see [2]). In recent times, a number of authors [3], [4] have noted that other irrational valued series may be constructed by replacing $n!$ in the series for e by the product $v_1v_2\dots v_n$, where $\{v_n\}$ is a strictly monotone increasing sequence of positive integers. However, in such cases, one needed to impose the additional assumption that $n \mid v_1v_2\dots v_n$ for each n. In this paper we shall demonstrate that irrational valued series may similarly be constructed from the terms of a generalized Fibonacci sequence, which are generated via the recurrence relation

$$U_n = PU_{n-1} - QU_{n-2},$$

where $P, Q \in \mathbf{Z}$ with $|P| > 1$, $|Q| = 1$, and $U_0 = 0$, $U_1 = 1$. The goal here is to establish the most general result possible by focusing attention on the following factorial-like expression

$$I(n) = U_kU_{k+1}\cdots U_{k+f(n)},$$

where $k \in \mathbf{N}\setminus\{0\}$ and $f: \mathbf{N} \to \mathbf{N}$ is an arbitrary strictly monotone increasing function. Such an expression will naturally reduce to the type of products considered above when $k = 1$ and $f(n) = n-1$. One advantage in dealing with the sequence $\{U_n\}$ is that we no longer need to impose the previous divisibility assumption, as this can be avoided by exploiting a fundamental property of generalized Fibonacci sequences that concerns the occurrence of a given prime factor in the sequence $\{U_n\}$. Unfortunately, the application of this property together with the argument used will require us to restrict the values of the ordered pairs (P, Q) to those prescribed above. To prove the desired result, we will employ here (as in [3]) an argument similar to that used by Euler in establishing the irrationality of e. However, before reaching this point, it will be necessary in Section 2 to acquaint ourselves with a few preliminary results, beginning with the aforementioned property of generalized Fibonacci sequences.

2. MAIN RESULT

In establishing the irrationality of the series in question, we shall need to invoke within our argument the following technical result: For any given $m \in \mathbf{N}\setminus\{0\}$, there exists a positive integer $N(m) > 0$ such that $m \mid I(n)$ whenever $n \geq N(m)$. This result, which holds irrespectively of the choice of k and $f(n)$, can be deduced directly from a divisibility property of generalized Fibonacci sequences. In order to state this property succinctly, we shall employ a number-theoretic function $\lambda_{rs}(n)$ as introduced in [1], which is defined below.

Let rs and $r+s$ be the roots of any quadratic equation of the form $x^2-ux+v=0$, where u and v are integers. Noting by the Symmetric Function Theorem that $(r-s)^{p-1}$ is an integer for an odd prime p, define the symbol $(\frac{r,s}{p})$ by the congruence

$$(r-s)^{p-1} \equiv \left(\frac{r,s}{p}\right) \bmod p,$$

where it is understood that $(\frac{r,s}{p})$ is the residue of least absolute value; whence $(\frac{r,s}{p}) = 0, +1$, or -1 according as $(r-s)^{p-1}$ is divisible by p, is a quadratic residue of p, or is a quadratic non-residue of p. In the case $p=2$, the symbol $(\frac{r,s}{2})$ is defined by:

$$\left(\frac{r,s}{2}\right) = \begin{cases} 1 & \text{if } rs \text{ is even,} \\ 0 & \text{if } rs \text{ is odd and } r+s \text{ is even,} \\ -1 & \text{if } rs \text{ and } r+s \text{ are both odd.} \end{cases}$$

Now, if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the different prime factors of n, define the functional value of $\lambda_{rs}(n)$ as the least common multiple of the numbers

$$p_i^{\alpha_1-1}\left[p_i - \left(\frac{r,s}{p_i}\right)\right], \; i = 1, 2, \ldots, k.$$

The important divisibility property that appeared as Theorem XIII in [1] can now be stated as follows.

Theorem 2.1: Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (P, Q). If the number $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are the different prime factors of n, is relatively prime to $rs = Q$ and if $\lambda = \lambda_{rs}(n)$, then $U_\lambda \equiv 0 \bmod n$.

It is clear from Theorem 2.1, when $\{U_n\}$ is generated with respect to the relatively prime pair (P, Q) with $|Q| = 1$, that given any $m \in \mathbf{N}\setminus\{0\}$ the sequence contains an element divisible by m. This fact does not allow us automatically to deduce the above technical result, since the product $I(n)$ in certain cases may never contain the term U_λ [i.e., when $\lambda_Q(m) < k$]. To deal with situations such as these, observe first from the above definition of λ_{rs} that

$$\lambda_Q(n) \geq p_i^{\alpha_i-1}\left[p_i - \left(\frac{r,s}{p_i}\right)\right] \geq p_i^{\alpha_i-1}[p_i-1] \geq p_i^{\alpha_i-1}.$$

Now, if given any positive integer t that contains a prime factor $p_i \geq k$ with $\alpha_i \geq 2$, then $\lambda_Q(tm) \geq p_i^{\alpha_i-1} \geq p_i \geq k$. So, provided that $(tm, Q) = 1$, the product $I(n)$ will contain, for n large, a term divisible by tm, namely $_{\lambda_Q(tm)}$. Thus, we can guarantee in the present case, as $|Q| = 1$, that $I(n)$ is divisible by m for suitably chosen n. Indeed, if one formally sets

$$t(m) = \min\{t \in \mathbf{N}\setminus\{0\} : \lambda_Q(tm) \geq k\},$$

then it is clear that $m \mid I(n)$ for $n \geq N(m)$, where

$$N(m) = \min\{n \in \mathbf{N}\setminus\{0\} \mid f(n) + k \geq \lambda_Q(t(m)m)\}.$$

Having established the required divisibility property of $I(n)$, we now need only introduce two further preliminary results before reaching the main theorem of this section. Both of these results will be used throughout the main argument of Theorem 2.2.

Lemma 2.1: Suppose $f:\mathbf{N}\to\mathbf{N}$ is a strictly monotone increasing function, then for all $r, m \in \mathbf{N}$, we have $f(m+r)-f(m)\geq r$.

Proof: Due to the strict monotonicity of *f*, it is clear that $f(m+i)-f(m+i-1)\geq 1$ for $i=1,2,\ldots,r$. Adding these r inequalities together and noting that the left-hand side is a telescoping sum equal to $f(m+r)-f(r)$, one deduces the desired inequality. □

Lemma 2.2: Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (*P*, *Q*), where $|P|>1$ and $|Q|=1$. Then the terms $|U_n|$ form a strictly monotone increasing sequence of integers.

Proof: We argue using induction. Clearly $|U_2|-|U_1|=|P|-1>0$. Now suppose the result holds for an integer $n=m>1$, that is, $|U_m|>|U_{m-1}|$. Now, by an application of the reverse triangle inequality, observe that

$$\begin{aligned}|U_{m+1}|-|U_m| &= |PU_m-QU_{m-1}|-|U_m|\\ &\geq |P||U_m|-|U_{m-1}|-|U_m|\\ &= (|P|-1)|U_m|-|U_{m-1}|>0,\end{aligned}$$

noting here that the final inequality follows from the inductive assumption and the fact that $|P|-1\geq 1$. Consequently, the result holds for $n=m+1$. □

Remark 2.1: With the above restrictions placed on the values of the ordered pairs (*P*, *Q*), it is clear from Lemma 2.2 that $I(n)\neq 0$ for $n\geq 1$. Thus, the terms of the series in question are well defined.

Theorem 2.2: Let $\{a_n\}$ be a bounded sequence of integers with the property that $a_n\neq 0$ for infinitely many *n*. Suppose further that $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (*P*, *Q*), where $|P|>1$ and $|Q|=1$. If $I(n)=U_kU_{k+1}\ldots U_{k+f(n)}$, where $k\in\mathbf{N}\setminus\{0\}$ and $f:\mathbf{N}\to\mathbf{N}$ is a strictly monotone increasing function, then the resulting series $\sum_{n=1}^{\infty}a_n/I(n)$ converges to an irrational sum.

Proof: We first establish convergence of the series. Observe that by setting $r=n-1$ for $n\in\mathbf{N}\setminus\{0\}$ and $m=1$ in Lemma 2.1, we have $f(n)\geq n-1+f(1)\geq n-1$. If $k\neq 1$, then each of the $f(n)+1$ terms in the product $|I(n)|$ are by Lemma 2.2 greater than or equal to $|U_2|=|P|$. Thus, $|I(n)|\geq |P|^{f(n)+1}\geq |P|^n$, while, if $k=1$ we have $|I(n)|=|U_2||U_3|\cdots|U_{1+f(n)}|\geq|P|^{f(n)}\geq |P|^{n-1}$. In any case, we have $|a_n|/|I(n)|\leq D/|P|^{n-1}$, where D is the upper bound for $\{a_n\}$; consequently, the series is absolutely convergent.

Suppose now, to the contrary, that the sum of the series is a rational number given by A/B, where $A,B\in\mathbf{Z}$ with $B\neq 0$. By the above technical result, there exists an $N(|B|)>0$, with $B\,|\,I(m)$ whenever $m\geq N(|B|)$. Choose $m\geq N(|B|)$ such that $D+1<|U_{k+f(m)+1}|$ and consider the following equality,

$$I(m)\frac{A}{B} - I(m)\sum_{n=1}^{m}\frac{a_n}{I(n)} = \sum_{n=m+1}^{\infty} a_n \frac{I(m)}{I(n)} = C. \tag{1}$$

Since by definition $I(n) | I(m)$ for $n \le m$, it is clear that $C \in \mathbf{Z}$. We now determine upper and lower bounds for $|C|$. First, note that the modulus of $I(m)/I(n)$ for $n \ge m+1$ in the series on the right of (1) is given by

$$\left|\frac{I(m)}{I(m+r)}\right| = (|U_{k+f(m)+1} \cdots U_{k+f(m+r)}|)^{-1},$$

where $r = 1, 2, \ldots$. Now, by Lemma 2.2, each of the $f(m+r) - f(m)$ terms in the denominator of the above expression are in modulus greater than or equal to $|U_{k+f(m)+1}|$, so by Lemma 2.1,

$$|U_{k+f(m)+1} \cdots U_{k+f(m+r)}| \ge |U_{k+f(m)+1}|^{f(m+r)-f(m)} \ge |U_{k+f(m)+1}|^{r}. \tag{2}$$

Hence, using the triangle inequality and (2), we have

$$\begin{aligned}|C| = \left|\sum_{r=1}^{\infty} a_{m+r}\frac{I(m)}{I(m+r)}\right| &\le \sum_{r=1}^{\infty} D\left|\frac{I(m)}{I(m+r)}\right| \\ &\le \sum_{r=1}^{\infty}\frac{D}{|U_{k+f(m)+1}|^{r}} = \frac{D}{|U_{k+f(m)+1}| - 1} < 1,\end{aligned}$$

noting here that the last inequality follows from our initial choice of m. To obtain a lower bound for $|C|$ set $p = \min\{n \ge m+1 : a_n \ne 0\}$ so $|a_p| \ge$. Then, by an application of the triangle and reverse triangle inequality, observe that

$$\begin{aligned}|C| = \left|\sum_{r=0}^{\infty} a_{p+r}\frac{I(m)}{I(p+r)}\right| &\ge \left| |a_p| \left|\frac{I(m)}{I(p)}\right| - \left|\sum_{r=1}^{\infty} a_{p+r}\frac{I(m)}{I(p+r)}\right| \right| \\ &\ge \left|\frac{I(m)}{I(p)}\right| - \sum_{r=1}^{\infty} D\left|\frac{I(m)}{I(p+r)}\right| = J.\end{aligned}$$

Clearly, from the definition, $p \ge m+1 > m$, thus, as in (2), we have for each $r \ge 1$,

$$\begin{aligned}\left|\frac{I(p)}{I(p+r)}\right| &= (|U_{k+f(p)+1} \cdots U_{k+f(p+r)}|)^{-1} \\ &\le \frac{1}{|U_{k+f(p)+1}|^{r}} < \frac{1}{|U_{k+f(m)+1}|^{r}}.\end{aligned}$$

Consequently,

$$\begin{aligned}|C| \ge J = \left|\frac{I(m)}{I(p)}\right|\left\{1 - \sum_{r=1}^{\infty} D\left|\frac{I(p)}{I(p+r)}\right|\right\} &\ge \left|\frac{I(m)}{I(p)}\right|\left\{1 - \sum_{r=1}^{\infty}\frac{D}{|U_{k+f(m)+1}|^{r}}\right\} \\ &= \left|\frac{I(m)}{I(p)}\right|\left\{1 - \frac{D}{|U_{k+f(m)+1}| - 1}\right\} > 0,\end{aligned}$$

where again the last inequality follows from the initial choice of m. Therefore, we have produced a $C \in \mathbf{Z}$ such that $0 < |C| < 1$, this obvious contradiction implies that the original assumption is false. Hence, the sum of the series in question is irrational. □

Remark 2.2: It was noted in [1] that no simple analog of Theorem 2.1 exists for the sequence of generalized Lucas numbers; thus, the above argument cannot readily be extended to establish a similar result involving these number sequences.

We now consider a simple consequence of Theorem 2.2.

Corollary 2.1: The base of the natural logarithm e is a nonalgebraic number of degree two.

Proof: Suppose that there exist $a, b, c \in \mathbf{Z}$ with $a \neq 0$ such that $ae^2 + be + c = 0$; then

$$ae + ce^{-1} = -b. \tag{3}$$

Now set $(P, Q) = (2, 1)$, $k = 1$, and $f(n) = n$ in Theorem 2.2. If $a_n = a + c(-1)^{n+1}$ we deduce, as $U_n = n$, that

$$\sum_{n=1}^{\infty} \frac{a_n}{I(n)} = a\sum_{n=1}^{\infty} \frac{1}{(n+1)!} + c\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} = a(e-2) + ce^{-1}$$

is an irrational number. Consequently, the number on the left of (3) is also irrational while the number on the right is clearly rational. This obvious contradiction thus establishes the above result. □

In view of Theorem 2.2, one may suspect that a similar result may hold for such factorial-like expressions as $I(n) = U_{f(n)} \cdots U_{f(n)+k}$, where $k \in \mathbf{N}$ and $f : \mathbf{N} \to \mathbf{N} \setminus \{0\}$ is a strictly monotone increasing function. At present, the author has been unable to supply an argument establishing, or a counterexample refuting, this conjecture. However, in the case of $f(n) = 2^n$ and $k = 0$, the author has been able to verify the irrationality of the series sum by direct calculation. To conclude, we now outline the derivation of the sum of these series.

Proposition 2.1: Suppose $\{U_n\}$ is a generalized Fibonacci sequence generated with respect to the relatively prime pair (P, Q), where $P \geq 1$, $Q = -1$; then

$$\sum_{n=1}^{\infty} \frac{1}{U_{2^n}} = \frac{P^2 + 4 - P\sqrt{P^2+4}}{2P}.$$

Proof: Consider the following telescoping sum

$$\sum_{n=1}^{N} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \sum_{n=1}^{N} \left(\frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}} \right)$$
$$= \frac{1}{1 - x^2} - \frac{1}{1 - x^{2^{N+1}}}.$$

If $|x| > 1$, then the above partial sums tend to a finite limit given by

$$\sum_{n=1}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{1}{1 - x^2}.$$

Now $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where α and β are the roots of $x^2 - Px - 1 = 0$. Consequently, as $\alpha\beta = -1$, we have, for $|\beta| > 1$,

$$\frac{1}{\alpha - \beta} \sum_{n=1}^{\infty} \frac{1}{U_{2^n}} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2^n} - \beta^{2^n}} = \sum_{n=1}^{\infty} \frac{\beta^{2^n}}{1 - \beta^{2^{n+1}}} = \frac{1}{1 - \beta^2}.$$

By setting $\alpha = (P - \sqrt{P^2+4})/2$ and $\beta = (P + \sqrt{P^2+4})/2$ in the above (noting that $|\alpha| < 1$ and $|\beta| > 1$) one obtains the desired sum. Note here that the irrationality of the series sum follows from the presence of the term $\sqrt{P^2+4}$, since P^2+4 is never a perfect square for $|P| \geq 1$. $\square$

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AMS Classification Numbers: 11B39, 40A05

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