SOME REMARKS ON FIBONACCI MATRICES

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In [1], Dazheng studies Fibonacci matrices, namely matrices M such that every entry of every positive power of M is either 0 or plus or minus a Fibonacci number. He gives 40 such four-by-four matrices. In the following, we give an interpretation of these matrices, from which we give simpler proofs of several of his theorems. We also determine all two-by-two Fibonacci matrices.

Let $\zeta = e^{2\pi i/5}$ be a primitive fifth root of unity. Then ζ is a root of the irreducible polynomial $X^4+X^3+X^2+X+1$, so the field $\mathbb{Q}(\zeta)$ is a vector space of dimension 4 over $\mathbb{Q}$ with basis $B=\{1,\zeta,\zeta^2,\zeta^3\}$. The ring of algebraic integers in $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$. The units of this ring are of the form $(-\zeta)^m\phi^n$, $0\le m\le 9$, $n\in\mathbb{Z}$, where $\phi=(1+\sqrt{5})/2=-(\zeta^2+\zeta^{-2})$.

If $\alpha\in\mathbb{Q}(\zeta)$, then multiplication by α gives a linear transformation of $\mathbb{Q}(\zeta)$, regarded as a vector space over $\mathbb{Q}$, and hence a matrix $M(\alpha)$ with respect to the basis B. For example, let $\alpha=\phi=-(\zeta^2+\zeta^3)$. Then

$$\begin{aligned}
\phi\cdot 1 &= -1\cdot\zeta^2-1\cdot\zeta^3,\\
\phi\cdot\zeta &= -\zeta^3-\zeta^4 = 1\cdot 1+1\cdot\zeta+1\cdot\zeta^2,\\
\phi\cdot\zeta^2 &= -\zeta^4-1 = 1\cdot\zeta+1\cdot\zeta^2+1\cdot\zeta^3,\\
\phi\cdot\zeta^3 &= -1\cdot 1-1\cdot\zeta.
\end{aligned}$$

Therefore,

$$M(\phi)=\begin{pmatrix}0&1&0&-1\\0&1&1&-1\\-1&1&1&0\\-1&0&1&0\end{pmatrix}.$$

This is the transpose of the matrix $\overline{F}_{10}$ of [1]. Similarly, we have the following matrices:

$$M(\zeta\phi)=\begin{pmatrix}1&0&-1&0\\1&1&-1&-1\\1&1&0&-1\\0&1&0&0\end{pmatrix},\quad M(\zeta^2\phi)=\begin{pmatrix}0&-1&0&0\\1&-1&-1&0\\1&0&-1&-1\\1&0&0&-1\end{pmatrix},\quad M(\zeta^3\phi)=\begin{pmatrix}-1&0&0&1\\-1&-1&0&1\\0&-1&-1&1\\0&0&-1&0\end{pmatrix},$$

$$M(\zeta^4\phi)=\begin{pmatrix}0&0&1&0\\-1&0&1&1\\-1&1&0&0\\-1&0&1&-1\end{pmatrix},\quad M(\phi^{-1})=\begin{pmatrix}-1&1&0&-1\\0&0&1&-1\\-1&1&0&0\\-1&-1&0&-1\end{pmatrix},\quad M(\zeta\phi^{-1})=\begin{pmatrix}1&0&-1&1\\0&1&-1&0\\1&0&0&0\\0&1&-1&1\end{pmatrix},$$

$$M(\zeta^2\phi^{-1})=\begin{pmatrix}0&-1&1&-1\\1&-1&0&0\\0&0&0&-1\\1&-1&1&-1\end{pmatrix},\quad M(\zeta^3\phi^{-1})=\begin{pmatrix}-1&1&-1&1\\-1&0&0&0\\0&0&-1&1\\-1&1&-1&0\end{pmatrix},\quad M(\zeta^4\phi^{-1})=\begin{pmatrix}1&-1&1&0\\0&0&0&1\\0&-1&1&0\\1&-1&0&1\end{pmatrix}.$$

In the notation of [1], these are the transposes of the matrices $\overline{F}_{20}$, $\overline{F}_{14}$, $\overline{F}_3$, $\overline{F}_8$, $\overline{F}_4$, $\overline{F}_{16}$, $\overline{F}_{12}$, $\overline{F}_2$, and $\overline{F}_{15}$, respectively. Letting $\overline{F}_{21-i}=-\overline{F}_i$ gives a set of 20 matrices corresponding to the numbers

$\pm\zeta^m\phi^n$, $0 \le m \le 4$, $n = \pm 1$. Note that any one of these numbers (often called fundamental units), together with $-\zeta$, generates the group of units of $\mathbb{Z}[\zeta]$.

Various properties of the matrices $\overline{F_i}$ follow immediately from the above. The following four propositions can be proved by straightforward calculations, but it is perhaps more interesting to see "conceptual" proofs.

Proposition 1 (= Proposition 4 of [1]): Let $1 \le i \le 20$. There exists k such that $\overline{F_i}^{-1} = \overline{F_k}$.

Proof: Let $\overline{F_i}$ correspond to $\varepsilon = \pm\zeta^m\phi^n$. Let $\overline{F_k}$ correspond to $\varepsilon^{-1} = \pm\zeta^{-m}\phi^{-n}$. Then $\overline{F_i}\overline{F_k}$ corresponds to multiplication by $\varepsilon^{-1}\varepsilon = 1$, so $\overline{F_i}\overline{F_k} = I$. □

Proposition 2 (= Proposition 5 of [1]): Let $1 \le i \le 20$. Then $\det(\overline{F_i}) = 1$.

Proof: The determinant is the norm of the corresponding number (see [3]). It is well known that the norm of a unit (of the ring of algebraic integers) is ± 1. Since the norm of a number from $\mathbb{Q}(\zeta)$ can be expressed as a product of two numbers times the product of their complex conjugates, the norm must be nonnegative. Therefore, the norm of a unit is 1. Since the numbers $\pm\zeta^m\phi^n$ are units, the determinants of the corresponding matrices must be 1. □

Proposition 3 (= Proposition 6 of [1]): Let $1 \le i, j \le 20$. Then $\overline{F_i}\overline{F_j} = \overline{F_j}\overline{F_i}$.

Proof: Multiplication in $\mathbb{Q}(\zeta)$ is commutative; therefore, multiplication of the corresponding matrices is commutative. □

Define the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

An easy calculation shows that A is the transpose of $M(-\zeta^4)$. Note that the powers of $-\zeta^4$ give all ten tenth roots of unity in $\mathbb{Q}(\zeta)$.

Proposition 4 (= Proposition 7 of [1]): Let $\mathcal{F}_1 = \{\overline{F_k} \mid k = 1, 3, 7, 8, 10, 11, 13, 14, 18, 20\}$ and let $\mathcal{F}_2 = \{\overline{F_k} \mid k = 2, 4, 5, 6, 9, 12, 15, 16, 17, 19\}$.

(a) Let $i = 1$ or 2. Given $\overline{F_h}, \overline{F_k} \in \mathcal{F}_i$, there exists $\overline{F_n} \in \mathcal{F}_i$ such that $\overline{F_h}\overline{F_k} = \pm\overline{F_n}^2$.

(b) If $\overline{F_h} \in \mathcal{F}_1$ and $\overline{F_k} \in \mathcal{F}_2$, then there exists n such that $\overline{F_h}\overline{F_k} = A^n$.

(c) Let $i = 1$ or 2. If $\overline{F_h}, \overline{F_k} \in \mathcal{F}_i$, then $\overline{F_h}^{10n} = \overline{F_k}^{10n}$ for all $n \in \mathbb{Z}$.

Proof: The matrices in $\mathcal{F}_1$ correspond to numbers of the form $\pm\zeta^m\phi$ and those in $\mathcal{F}_2$ correspond to numbers of the form $\pm\zeta^m\phi^{-1}$. The properties of the matrices now follow from the form of these numbers. □

We now come to the main theorem. It was proved in [1] by fixing indices $1 \le h \le 20$ and $0 \le i \le 9$ and expressing the entries of $\overline{F_h}^{10k+i}$ in terms of Fibonacci numbers of the form $\pm F_{ak+b}$ or 0 for $k = 0, 1, 2, \ldots$. This gives the additional information that, for each index h, each Fibonacci number occurs in $\overline{F_h}^n$ for some n (in fact, this property was included in the definition of a Fibonacci matrix in [1]). With a little more care, this can be deduced from the following proof.

Theorem 1 (= Proposition 1 of [1]): Let $1 \le h \le 20$ and let n be a positive integer. Every entry of $\overline{F}_h^n$ is either 0 or $\pm F_m$ for some Fibonacci number F_m, where $m = n-1, n,$ or $n+1$.

Proof: Fix $n \ge 1$. For each a mod 5, let

$$g_n(a) = \sum_{\substack{j=0 \\ j \equiv a \pmod 5}}^{n} \binom{n}{j}.$$

Lemma 1: $5g_n(a) = \sum_{i=0}^4 \zeta^{-ai}(1+\zeta^i)^n$.

Proof: The right side is

$$\sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{4} \zeta^{i(j-a)}.$$

Since $\sum_{i=0}^4 \zeta^{ib} = 0$ when $b \not\equiv 0 \pmod 5$ and equals 5 when $b \equiv 0 \pmod 5$, the result follows. □

Lemma 2: For any values of a and b, the difference $g_n(a) - g_n(b)$ is either 0 or $\pm F_m$ for some Fibonacci number F_m, where $m = n-1, n,$ or $n+1$.

Proof: Using the fact that $1+\zeta = -\zeta^{-2}\phi$, $1+\zeta^2 = \zeta\phi^{-1}$, $1+\zeta^3 = \zeta^{-1}\phi^{-1}$, and $1+\zeta^4 = -\zeta^2\phi$, we find that

$$\begin{aligned} 5g_n(a) - 5g_n(b) &= \sum_{i=0}^{4} \zeta^{-ai}(1+\zeta^i)^n - \sum_{i=0}^{4} \zeta^{-bi}(1+\zeta^i)^n \\ &= (-\phi)^n(\zeta^{a+2n} + \zeta^{-a-2n} - \zeta^{b+2n} - \zeta^{-b-2n}) \\ &\quad + (\phi^{-1})^n(\zeta^{n-2a} + \zeta^{-n+2a} - \zeta^{n-2b} - \zeta^{-n+2b}). \end{aligned}$$

Since $a + 2n \equiv 2(n-2a) \pmod 5$, we find that we have the following cases:

(1) $\zeta^{a+2n} + \zeta^{-a-2n} = \zeta + \zeta^{-1} = \phi^{-1}$ and $\zeta^{n-2a} + \zeta^{-n+2a} = \zeta^2 + \zeta^3 = -\phi$,

(2) $\zeta^{a+2n} + \zeta^{-a-2n} = \zeta^2 + \zeta^3 = -\phi$ and $\zeta^{n-2a} + \zeta^{-n+2a} = \zeta + \zeta^{-1} = \phi^{-1}$,

(3) $\zeta^{a+2n} + \zeta^{-a-2n} = 2$ and $\zeta^{n-2a} + \zeta^{-n+2a} = 2$.

Similarly, we have three cases for the terms involving b.

The coefficient of $(-\phi)^n$ is therefore 0 or one of the following:

(a) $\pm(\phi^{-1} - (-\phi)) = \pm\sqrt{5}$,

(b) $\pm(\phi^{-1} - 2) = \mp\sqrt{5}\phi^{-1}$,

(c) $\pm(-\phi - 2) = \mp\sqrt{5}\phi$.

The corresponding coefficients of ϕ^{-n} are 0 and $\mp\sqrt{5}, \mp\sqrt{5}\phi$, and $\mp\sqrt{5}\phi^{-1}$, respectively.

Putting everything together, we find that $5g_n(a) - 5g_n(b)$ is, up to sign, either 0 or one of the following:

$$\begin{aligned} \sqrt{5}((-\phi)^n - \phi^{-n}) &= (-1)^n 5F_n, \\ \sqrt{5}((-\phi)^{n-1} - \phi^{-n+1}) &= (-1)^{n-1} 5F_{n-1}, \\ \sqrt{5}((-\phi)^{n+1} - \phi^{-n-1}) &= (-1)^{n-1} 5F_{n+1}. \end{aligned}$$

This proves the lemma. □

We can now prove Theorem 1. The matrix $\overline{F}_h^n$ corresponds to a number of the form $(\pm\zeta^m\phi^{\pm 1})^n$, which is of the form $\pm\zeta^a(1+\zeta)^n$ or of the form $\pm\zeta^a(1+\zeta^2)^n$. We may ignore the $\pm$.

Consider first $\zeta^a(1+\zeta)^n$. We must multiply this times a power of ζ and then express the result as a linear combination of elements of the basis B. Since the exponent a is already arbitrary, we need only show that when we express a number of the form $^a(1+\zeta)^n$ in terms of B the coefficients are Fibonacci numbers (up to sign) or 0. By the binomial theorem, we have

$$\zeta^a(1+\zeta)^n = \sum_{j=0}^{n}\binom{n}{j}\zeta^{j+a} = \sum_{i=0}^{4} g_n(i-a)\zeta^i = \sum_{i=0}^{3}(g_n(i-a)-g_n(4-a))\zeta^i.$$

Lemma 2 yields the result in this case.

Now consider $\zeta^a(1+\zeta^2)^n$, which equals

$$\sum_{j=0}^{n}\binom{n}{j}\zeta^{2j+a} = \sum_{i=0}^{4} g_n(3i-3a)\zeta^i \quad [\text{since } 2j+a \equiv i \pmod 5 \Rightarrow j \equiv 3i-3a]$$

$$= \sum_{i=0}^{3}(g_n(3i-3a)-g_n(2-3a))\zeta^i.$$

The result again follows from Lemma 2. □

The Two-by-Two Case

Theorem 2: Let M be a two-by-two matrix such that each entry of M^n for $n=1,2,3,\ldots$ is either 0 or plus or minus a Fibonacci number. Suppose in addition that not all of the entries of M^n are bounded as $n\to\infty$. Then $\pm M$ is a power of one of the following matrices:

$$\begin{pmatrix}0 & \pm 1\\ \pm 1 & 1\end{pmatrix},\ \begin{pmatrix}1 & \pm 1\\ \pm 1 & 0\end{pmatrix},\ \begin{pmatrix}2 & \pm 1\\ \mp 1 & -1\end{pmatrix},\ \begin{pmatrix}-1 & \pm 1\\ \mp 1 & 2\end{pmatrix}.$$

Remark: It is well known, and will follow from the proof of the theorem, that

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^n = \begin{pmatrix}F_{n-1} & F_n\\ F_n & F_{n+1}\end{pmatrix}$$

and that

$$\begin{pmatrix}-1 & -1\\ 1 & 2\end{pmatrix}^n = \begin{pmatrix}(-1)^n F_{n-2} & -F_n\\ F_n & F_{n+2}\end{pmatrix}.$$

From the point of view used above, the first matrix arises from multiplication by ϕ with respect to the basis $\{1,\phi\}$ of $\mathbb{Q}(\sqrt{5})$, and the second matrix arises from multiplication by ϕ with respect to the basis $\{1,\phi^2\}$.

Proof: We start with the following.

Lemma 3: Suppose a_n, $n=1,2,\ldots$, is a sequence of nonzero integers such that each a_n is plus or minus a Fibonacci number and such that $\lambda = \lim a_{n+1}/a_n$ exists. Then λ is of the form $\pm\phi^r$ for some integer $r\geq 0$. If the sequence a_n is unbounded, $r\geq 1$.

Proof: Let $a_n = \pm F_{m_n}$. The limit λ cannot be of absolute value less than 1 since the a_n are integers. Clearly, $\lambda=\pm 1$ is equivalent to the boundedness of a_n, so henceforth assume the sequence a_n is unbounded. It follows easily that $\lim F_{m_n} = \infty$, hence $\lim m_n = \infty$. Therefore, $\lim F_{m_n} / \phi^{m_n} = 1/\sqrt{5}$, so

$$|\lambda| = \lim \frac{F_{m_{n+1}} \phi^{m_n}}{F_{m_n} \phi^{m_{n+1}}} \phi^{m_{n+1}-m_n} = \lim \phi^{m_{n+1}-m_n}.$$

Since the powers of ϕ are discrete in the positive reals, $m_{n+1} - m_n$ must eventually be constant, say r. Since $m_n \to \infty$, $r \geq 1$. This proves the lemma. □

Since the elements of the powers of a matrix satisfy a second-order recursion, we need the following result. Recall that we can define Fibonacci numbers for negative indices by $F_{-n} = (-1)^{n+1} F_n$.

Lemma 4: Let $a_1, a_2, \ldots$ be an unbounded sequence of integers satisfying a second-order linear recursion with constant coefficients: $a_{n+2} = ua_{n+1} + va_n$. Suppose each a_n is either 0 or $\pm F_{m_n}$ for some Fibonacci numbers F_{m_n}. Then there are integers r and s (possibly negative) and a choice $\delta = \pm 1$ of sign, independent of n, such that $a_n = \delta F_{rn+s}$ for all n (we allow Fibonacci numbers with negative indices; see above).

Remark: This result follows, for example, from work of van der Poorten (see the remarks at the end of this article). However, it seems reasonable to give a self-contained proof.

Proof: We have not assumed that the coefficients u, v of the recursion are rational numbers, so we first show that this must be the case. The recursion shows that each vector (a_{n+1}, a_n) is a linear combination of (a_2, a_1) and (a_3, a_2). Suppose $\det\begin{pmatrix} a_2 & a_1 \\ a_3 & a_2 \end{pmatrix} = 0$. If $a_1 = 0$, then $a_2 = 0$, so $a_n = 0$ for all n, contrary to our assumptions. Therefore, assume $a_1 \neq 0$. Then all these vectors are multiples of (a_2, a_1), which implies that

$$a_n = a_1 \left(\frac{a_2}{a_1} \right)^{n-1} \tag{1}$$

for all $n \geq 1$. Therefore, $|a_n| \to \infty$ (otherwise $|a_2 / a_1| \leq 1$ and the sequence is bounded) and $a_{n+1} / a_n = a_2 / a_1$. Since $|a_n|$ is a Fibonacci number [it cannot be 0 by (1)], Lemma 3 implies that $a_2 / a_1 = \pm \phi^r$ for some $r \geq 1$. Since all positive powers of ϕ are irrational, this is impossible. This contradiction shows that the determinant is nonzero.

Since

$$\begin{pmatrix} a_2 & a_1 \\ a_3 & a_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix},$$

and the matrix is invertible, the rationality of a_1, a_2, a_3, a_4 implies that u and v are rational.

Remark: The recursion $a_{n+2} = \pi a_{n+1} + (4 - 2\pi) a_n$, which is satisfied by the rational numbers $a_n = 2^n$, shows that the rationality of the numbers a_n is not sufficient to guarantee that u, v are rational.

Let α and β be the two roots of $X^2 - uX - v$. If $\alpha \neq \beta$, then there are constants A and B such that $a_n = A\alpha^n + B\beta^n$. There are several cases to consider, depending on the relative magnitudes of α and β.

Case 1. $|\alpha| > |\beta|$

If $A = 0$, we can replace the pair (α, β) by $(\beta, 0)$ in the following argument (and eventually conclude that $A \neq 0$). Therefore, assume $A \neq 0$, so $\lim a_{n+1} / a_n = \alpha$. Since the sequence is unbounded, $|\alpha| > 1$, so $a_n \neq 0$ for all sufficiently large *n*, and each a_n is plus or minus a Fibonacci number. By Lemma 3, $\alpha = \pm\phi^r$ for some $r \geq 1$. Therefore, α is irrational, so the polynomial $X^2 - uX - v$ is irreducible in $\mathbb{Q}[X]$. Since β is also a root, it must be the conjugate $\pm(-\phi)^{-r}$ of α.

Let $\sigma = \text{sign}(\alpha)$, so $\alpha = \sigma\phi^r$. Let δ_n be the sign of a_n. Note that

$$A = \lim \frac{a_n}{(\sigma\phi^r)^n}.$$

This implies that $\delta_n = \text{sign}(A)\sigma^n$ for *n* sufficiently large. Also, $a_n = \delta_n F_{m_n}$, so

$$\lim \frac{\delta_n a_n}{\phi^{m_n}} = \frac{1}{\sqrt{5}}.$$

Therefore,

$$A = \lim \frac{\delta_n a_n}{\phi^{m_n}} \delta_n \sigma^{-n} \phi^{m_n - nr} = \frac{\text{sign}(A)}{\sqrt{5}} \lim \phi^{m_n - nr}.$$

Since the powers of ϕ are discrete, eventually $m_n - nr$ must stabilize: there exists $s \in \mathbb{Z}$ such that $m_n - nr = s$ for all sufficiently large *n*. This also yields $A = \pm\phi^s / \sqrt{5}$. Since the terms with ϕ^{rn} cancel in the equation

$$\begin{aligned} A(\sigma\phi^r)^n + B(\sigma(-\phi)^{-r})^n = a_n = \delta_n F_{m_n} &= \text{sign}(A)\sigma^n F_{rn+s} \\ &= \frac{\text{sign}(A)\sigma^n}{\sqrt{5}}(\phi^{rn+s} - (-\phi^{-1})^{rn+s}), \end{aligned}$$

it follows that $B = -\text{sign}(A)(-\phi)^{-s} / \sqrt{5}$. We have proved that $a_n = \pm(\pm 1)^n F_{rn+s}$. By changing the signs of *r*, *s* if necessary, we can absorb the $(\pm 1)^n$. This yields the result of the lemma in Case 1.

It remains to show that the other cases do not occur.

Case 2. $\alpha = -\beta$

In this case, $u = \alpha + \beta = 0$, so the recursion is $a_{n+2} = va_n$. Since the sequence is assumed to be unbounded, $|v| > 1$ and some $a_{n_0} \neq 0$. Therefore, $a_{n_0+2k+2} / a_{n_0+2k} = v^2 \in \mathbb{Q}$. Since the numbers $a_{n_0+2k} = a_{n_0} v^{2k}$ are nonzero, they are Fibonacci numbers up to sign. Lemma 3 implies that $v^2 = \phi^r$ for some $r \geq 1$. This is impossible.

Case 3. $\alpha = \beta$

In this case, $a_n = A\alpha^n + Bn\alpha^n$. Hence, $a_n \neq 0$ for sufficiently large *n*, and $\lim a_{n+1} / a_n = \alpha$. By Lemma 3, $\alpha = \pm\phi^r$ for some $r \geq 1$. Since $\alpha = u / 2 \in \mathbb{Q}$, this is impossible.

Case 4. $\overline{\alpha} = \beta$, $\alpha \neq \beta$

Since $a_n = A\alpha^n + B\overline{\alpha}^n \in \mathbb{Q}$ for all *n*, we must have $B = \overline{A}$. Write $A = Re^{i\gamma}$ and $\alpha = \rho e^{i\theta}$. Then

$$a_n = R\rho^n e^{in\theta + i\gamma} + R\rho^n e^{-in\theta - i\gamma} = 2R\rho^n \cos(n\theta + \gamma).$$

Suppose first that $\theta / 2\pi \notin \mathbb{Q}$. By a theorem of Weyl (see [2], Theorem 445), the sequence of fractional parts of $n\theta / 2\pi$ is uniformly distributed in the interval [0, 1]. In particular, there is a sequence of integers n_i such that $n_i\theta / 2\pi + (\theta + 2\gamma) / 4\pi - k_i$ is very small for some integers k_i, and the limit is 0 as $i \to \infty$. Therefore, $\cos(n_i\theta + \gamma)$ is very close to $\cos(2\pi k_i - \theta / 2) = \cos(-\theta / 2)$ and $\cos((n_i + 1)\theta + \gamma)$ is very close to $\cos(\theta / 2) = \cos(-\theta / 2)$. Therefore, $\lim a_{n_i+1} / a_{n_i} = \rho$. Lemma 3 shows that $\rho = \phi^r$ for some $r \geq 0$. But $v = \alpha\beta = \rho^2$, so $\phi^{2r} \in \mathbb{Q}$, which implies $r = 0$. Therefore, the sequence a_n is bounded, contrary to assumption.

Now suppose that $\theta / 2\pi = w / z \in \mathbb{Q}$, where $w, z \in \mathbb{Z}$. Choose n_0 such that $a_{n_0} \neq 0$. Then $a_{n_0+(k+1)z} / a_{n_0+kz} = \rho^z$ for $k = 0, 1, 2, \ldots$. Lemma 3 implies that $\rho^z = \phi^r$ for some $r \geq 0$. Therefore, $\phi^{2r} = \rho^{2z} = v^z \in \mathbb{Q}$, so $r = 0$, which is impossible.

It is easy to see that Cases 1-4 exhaust all possibilities for α, β. This concludes the proof of Lemma 4. □

Corollary: Suppose A, B, α, β are complex numbers such that for each $n \geq 1$ the number $a_n = A\alpha^n + B\beta^n$ is either 0 or plus or minus a Fibonacci number, and such that the sequence a_n is unbounded. Then there are integers $r \geq 1$ and s such that (assume $|\alpha| \geq |\beta|$)

$$\alpha = \pm\phi^r, \qquad \beta = \pm(-\phi)^{-r}$$

and

$$A = \pm\frac{\phi^s}{\sqrt{5}}, \qquad B = \mp\frac{(-\phi)^{-s}}{\sqrt{5}}.$$

Proof: This is a restatement of what was proved above, combined with the fact that the sequence a_n uniquely determines the numbers A, B, α, β. □

We can now prove Theorem 2. Suppose the matrix M is as in the statement of the theorem, and let α, β be the roots of the characteristic polynomial of M. The case $\alpha = \beta$ corresponds to Case 3 in the proof of Lemma 4, and the reasoning below shows that it cannot occur, so we assume $\alpha \neq \beta$. Then M is diagonalizable, so there are complex numbers a, b, c, and d with $ad = bc \neq 0$ such that

$$M = \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Therefore,

$$M^n = \frac{1}{ad - bc}\begin{pmatrix} ad\alpha^n - bc\beta^n & -ab(\alpha^n - \beta^n) \\ cd(\alpha^n - \beta^n) & ad\beta^n - bc\alpha^n \end{pmatrix}.$$

We assume $|\alpha| \geq |\beta|$. By the Corollary,

$$\alpha = \pm\phi^r, \quad \beta = \pm(-\phi)^{-r},$$

for some integer r. Since not all entries are bounded, $r \geq 1$. If $ad = 0$ then $bc \neq 0$; looking at the first entry in the matrix yields $\beta^n \in \mathbb{Z}$ for all n, which is impossible. Similarly, $bc \neq 0$. By the Corollary,

$$\frac{ad}{ad - bc} = \pm\frac{\phi^s}{\sqrt{5}} \quad \text{and} \quad \frac{-bc}{ad - bc} = \mp\frac{(-\phi)^{-s}}{\sqrt{5}}$$

for some integer s. Therefore,

$$1 = \frac{ad}{ad-bc} - \frac{bc}{ad-bc} = \pm\frac{\phi^s - (-\phi)^{-s}}{\sqrt{5}} = \pm F_s,$$

so $s = \pm 1, \pm 2$.

Consider the upper right corner of M^n. Since $ab \neq 0$, the only possibility allowed by the Corollary is $ab/(ad-bc) = \pm 1/\sqrt{5}$. Similarly, $cd/(ad-bc) = \pm 1/\sqrt{5}$. Therefore, $ab = \pm cd$.

Since the matrix $\left(\begin{smallmatrix} 1/a & 0 \\ 0 & 1/d \end{smallmatrix}\right)$ commutes with $\left(\begin{smallmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{smallmatrix}\right)$, we can replace the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/a & 0 \\ 0 & 1/d \end{smallmatrix}\right)$ and, therefore, assume $a = d = 1$. This makes the calculations simpler. We now have the following equations (the choices of signs are independent):

$$b = \pm c, \quad \frac{1}{bc} = \frac{ad/(ad-bc)}{bc/(ad-bc)} = (-\phi^2)^s, \quad s = \pm 1, \pm 2.$$

Since $\alpha, \beta \in \mathbb{Q}(\sqrt{5})$, the diagonalizing matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ may be assumed to have entries in $\mathbb{Q}(\sqrt{5})$. Therefore, the case $b = c$, $s = \pm 1$ and the case $b = -c$, $s = \pm 2$ cannot occur. Checking all solutions in the remaining cases and substituting into the formula for M shows that $\pm M$ is the r^{th} power of one of the matrices in the statement of the theorem. The same calculation yields that each entry of the powers of the matrices in the theorem is plus or minus a Fibonacci number. This completes the proof of Theorem 2. □

In [1], the problem is posed to find all four-by-four Fibonacci matrices. This can be attacked by the above method. One difficulty is proving the analog of Lemma 4 for fourth-order recurrences. A result of van der Poorten ([4], pp. 514-15) says that if an infinite sequence of elements $\{b_0, b_1, \ldots\}$ chosen from the members of a nondegenerate (i.e., no ratio of characteristic roots of the recurrence is a root of unity) recurrent sequence $\{a_0, a_1, \ldots\}$ again forms a recurrent sequence, then there is an integer $d > 0$, and a set R of integers r with $0 \le r < d$, such that for all h we have $b_h = a_{r_h + hd}$ and $r_h \in R$ is periodic mod d. Since the entries in the powers of a matrix form a recurrent sequence, and the Fibonacci numbers form a nondegenerate sequence, this result applies, and we find that the eigenvalues of the matrix must be roots of unity times powers of ϕ. This reduces the problem to the consideration of several cases for the characteristic roots.

The other difficulty is the calculation involving the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, since it must be replaced by a four-by-four matrix. The calculations are probably possible, but surely would be more difficult.

To conclude, we give a few more four-by-four Fibonacci matrices. They are not as good examples as $\overline{F}_1, \ldots, \overline{F}_{20}$ since they all have powers that are reducible. However, they indicate various possibilities that can arise. They were chosen using the fact that their eigenvalues must be roots of unity times powers of ϕ.

1. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

This is obtained by considering multiplication by $\zeta\phi$ on the basis $(\{1, \phi, \zeta, \zeta\phi\}$. This matrix is almost reducible in the sense that

$$M^5 = \begin{pmatrix} 3 & 5 & 0 & 0 \\ 5 & 8 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 5 & 8 \end{pmatrix}.$$

This of course can be predicted from the fact that $(\zeta\phi)^5 = \phi^5$.

2. The matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

is obtained from multiplication by $\zeta\phi$ on the basis $\{1, \phi^2, \zeta, \zeta\phi^2\}$. The fifth power of this matrix is reducible.

3. The matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

is obtained from multiplication by $\zeta_3\phi$ on the basis $\{1, \phi, \zeta_3, \zeta_3\phi\}$ of $\mathbb{Q}(\phi, \zeta_3)$, where ζ_3 is a primitive third root of unity. The third power of this matrix is reducible.

4. The matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 2 \\ -1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

is obtained from multiplication by $i\phi$ on the basis $\{1, \phi, i, i\phi\}$ of $\mathbb{Q}(\phi, i)$. More generally, any Fibonacci matrix tensored with a permutation matrix, in this case $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, will give a Fibonacci matrix.

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