

COMBINATORIAL SUMS AND SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

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0. INTRODUCTION

In this note we deal with several combinatorial sums and series involving inverses of binomial coefficients. Some of them have already been considered by other authors (see, e.g., [3], [4]), but it should be noted that our approach is different. It is based on Euler's well-known Beta function defined by

$$B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt$$

for all positive integers m and n . Since

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

we get

$$\binom{n}{k}^{-1} = (n+1) \int_0^1 t^k (1-t)^{n-k} dt \tag{1}$$

for all nonnegative integers n and k with $n \geq k$.

1. SUMS INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 1.1 ([4], Theorem 1): If n is a nonnegative integer, then

$$\sum_{k=0}^n \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.$$

Proof: Let S_n be the sum of inverses of binomial coefficients. From (1) we get

$$\begin{aligned} S_n &= \sum_{k=0}^n (n+1) \int_0^1 t^k (1-t)^{n-k} dt \\ &= (n+1) \int_0^1 \left\{ (1-t)^n \sum_{k=0}^n \left(\frac{t}{1-t} \right)^k \right\} dt = (n+1) \int_0^1 \frac{(1-t)^{n+1} - t^{n+1}}{1-2t} dt. \end{aligned}$$

Making the substitution $1-2t = x$, we obtain

$$\begin{aligned} S_n &= \frac{n+1}{2^{n+2}} \int_{-1}^1 \frac{(1+x)^{n+1} - (1-x)^{n+1}}{x} dx = \frac{n+1}{2^{n+2}} \left\{ \int_{-1}^1 \frac{(1+x)^{n+1} - 1}{x} dx + \int_{-1}^1 \frac{1 - (1-x)^{n+1}}{x} dx \right\} \\ &= \frac{n+1}{2^{n+2}} \sum_{k=0}^n \left\{ \int_{-1}^1 (1+x)^k dx + \int_{-1}^1 (1-x)^k dx \right\} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}. \end{aligned}$$

Theorem 1.2: If n is a positive integer, then

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n}{2k}^{-1} = \frac{4n+1}{2n+1}.$$

Proof: Formula (1) yields

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n}{2k}^{-1} &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (4n+1) \int_0^1 t^{2k} (1-t)^{4n-2k} dt \\ &= (4n+1) \int_0^1 \left\{ (1-t)^{4n} \sum_{k=0}^{2n} \binom{2n}{k} \left(\frac{-t^2}{(1-t)^2} \right)^k \right\} dt \\ &= (4n+1) \int_0^1 (1-t)^{4n} \left(1 - \frac{t^2}{(1-t)^2} \right)^{2n} dt \\ &= (4n+1) \int_0^1 (1-2t)^{2n} dt = \frac{4n+1}{2n+1}. \end{aligned}$$

Theorem 1.3 ([5]): If n is a positive integer, then

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} \binom{2n}{k}^{-1} = -\frac{1}{2n-1}.$$

Proof: Let S_n be the sum to evaluate. From (1) we get

$$\begin{aligned} S_n &= \sum_{k=0}^{2n} (-1)^k \binom{4n}{2k} (2n+1) \int_0^1 t^k (1-t)^{2n-k} dt \\ &= (2n+1) \int_0^1 \left\{ \sum_{k=0}^{2n} \binom{4n}{2k} (-1)^k t^k (1-t)^{2n-k} \right\} dt \\ &= \frac{2n+1}{2} \int_0^1 \left\{ (\sqrt{1-t} + i\sqrt{t})^{4n} + (\sqrt{1-t} - i\sqrt{t})^{4n} \right\} dt. \end{aligned}$$

Since

$$\sqrt{1-t} \pm i\sqrt{t} = \cos \left(\arctan \sqrt{\frac{t}{1-t}} \right) \pm i \sin \left(\arctan \sqrt{\frac{t}{1-t}} \right),$$

it follows that

$$S_n = (2n+1) \int_0^1 \cos \left(4n \arctan \sqrt{\frac{t}{1-t}} \right) dt.$$

Making the substitution $\arctan \sqrt{\frac{t}{1-t}} = x$, we obtain

$$\begin{aligned} S_n &= (2n+1) \int_0^{\pi/2} \cos(4nx) \sin(2x) dx \\ &= \frac{2n+1}{2} \int_0^{\pi/2} \{ \sin(4n+2)x - \sin(4n-2)x \} dx = -\frac{1}{2n-1}. \end{aligned}$$

Theorem 1.4 ([2]): If m , n , and p are nonnegative integers with $p \leq n$, then

$$\sum_{k=0}^m \binom{m}{k} \binom{n+m}{p+k}^{-1} = \frac{n+m+1}{n+1} \binom{n}{p}^{-1}.$$

Proof: Formula (1) yields

$$\begin{aligned} \sum_{k=0}^m \binom{m}{k} \binom{n+m}{p+k}^{-1} &= \sum_{k=0}^m \binom{m}{k} (n+m+1) \int_0^1 t^{p+k} (1-t)^{n+m-p-k} dt \\ &= (n+m+1) \int_0^1 \left\{ t^p (1-t)^{n+m-p} \sum_{k=0}^m \binom{m}{k} \left(\frac{t}{1-t} \right)^k \right\} dt \\ &= (n+m+1) \int_0^1 t^p (1-t)^{n+m-p} \left(1 + \frac{t}{1-t} \right)^m dt \\ &= (n+m+1) \int_0^1 t^p (1-t)^{n-p} dt = \frac{n+m+1}{n+1} \binom{n}{p}^{-1}. \end{aligned}$$

Remark: In the special case $p = n$, from the above theorem we get

$$\sum_{k=0}^m \binom{m}{k} \binom{n+m}{n+k}^{-1} = \frac{n+m+1}{n+1}.$$

Theorem 1.5: If m and n are nonnegative integers, then

$$\sum_{k=0}^n (-1)^k \binom{m+n}{m+k}^{-1} = \frac{m+n+1}{m+n+2} \left(\binom{m+n+1}{m}^{-1} + (-1)^n \right).$$

Proof: We have

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{m+n}{m+k}^{-1} &= \sum_{k=0}^n (-1)^k (m+n+1) \int_0^1 t^{m+k} (1-t)^{n-k} dt \\ &= (m+n+1) \int_0^1 \left\{ t^m (1-t)^n \sum_{k=0}^n \binom{n}{k} \left(\frac{-t}{1-t} \right)^k \right\} dt \\ &= (m+n+1) \left(\int_0^1 t^m (1-t)^{n+1} dt + (-1)^n \int_0^1 t^{m+n+1} dt \right) \\ &= \frac{m+n+1}{m+n+2} \left(\binom{m+n+1}{m}^{-1} + (-1)^n \right). \end{aligned}$$

Remark: In the special case $m = n$ we get

$$\sum_{k=0}^n (-1)^k \binom{2n}{n+k}^{-1} = \frac{2n+1}{2n+2} \left(\binom{2n+1}{n}^{-1} + (-1)^n \right),$$

while in the special case $m = 0$ we obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^{-1} = \frac{n+1}{n+2} (1 + (-1)^n).$$

Consequently (see [3], p. 343),

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^{-1} = \frac{2n+1}{n+1}.$$

Theorem 1.6: If n is a positive integer, then

$$\begin{aligned} & \frac{1}{3n+1} \sum_{k=0}^n \binom{3n}{3k}^{-1} - \frac{1}{3n+2} \sum_{k=0}^n \binom{3n+1}{3k+1}^{-1} + \frac{1}{3n+3} \sum_{k=0}^n \binom{3n+2}{3k+2}^{-1} \\ & = \frac{1}{3n+3} \sum_{k=0}^{3n+2} \binom{3n+2}{k}^{-1}. \end{aligned}$$

Proof: We have

$$\begin{aligned} & \frac{1}{3n+1} \sum_{k=0}^n \binom{3n}{3k}^{-1} - \frac{1}{3n+2} \sum_{k=0}^n \binom{3n+1}{3k+1}^{-1} + \frac{1}{3n+3} \sum_{k=0}^n \binom{3n+2}{3k+2}^{-1} \\ & = \sum_{k=0}^n \int_0^1 \{t^{3k}(1-t)^{3n-3k} - t^{3k+1}(1-t)^{3n-3k} + t^{3k+2}(1-t)^{3n-3k}\} dt \\ & = \int_0^1 \left\{ (1-t)^{3n}(1-t+t^2) \sum_{k=0}^n \left(\frac{t^3}{(1-t)^3} \right)^k \right\} dt = \int_0^1 \frac{(1-t)^{3n+3} - t^{3n+3}}{1-2t} dt. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{1}{3n+3} \sum_{k=0}^{3n+2} \binom{3n+2}{k}^{-1} & = \sum_{k=0}^{3n+2} \int_0^1 t^k (1-t)^{3n+2-k} dt \\ & = \int_0^1 \left\{ (1-t)^{3n+2} \sum_{k=0}^{3n+2} \left(\frac{t}{1-t} \right)^k \right\} dt = \int_0^1 \frac{(1-t)^{3n+3} - t^{3n+3}}{1-2t} dt, \end{aligned}$$

completing the proof.

2. SERIES INVOLVING INVERSES OF BINOMIAL COEFFICIENTS

Theorem 2.1: If m and n are positive integers with $m > n$, then

$$\sum_{k=0}^{\infty} \binom{mk}{nk}^{-1} = \int_0^1 \frac{1 + (m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2} dt.$$

Proof: From (1) we get

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{mk}{nk}^{-1} & = \sum_{k=0}^{\infty} (mk+1) \int_0^1 t^{nk} (1-t)^{(m-n)k} dt \\ & = m \sum_{k=1}^{\infty} \int_0^1 k (t^n(1-t)^{m-n})^k dt + \sum_{k=0}^{\infty} \int_0^1 (t^n(1-t)^{m-n})^k dt. \end{aligned}$$

Let $f : [0, 1] \rightarrow \mathbf{R}$ be the function defined by $f(t) = t^n(1-t)^{m-n}$. It is immediately seen that f attains its maximum at the point $t_0 = n/m$. Since $f(t_0) < 1$, it follows that

$$\sum_{k=1}^{\infty} k (t^n(1-t)^{m-n})^k = \frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2}$$

and

$$\sum_{k=1}^{\infty} (t^n(1-t)^{m-n})^k = \frac{1}{1-t^n(1-t)^{m-n}}$$

uniformly on $[0, 1]$. Therefore, we obtain

$$\sum_{k=0}^{\infty} \binom{mk}{nk}^{-1} = m \int_0^1 \frac{t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2} dt + \int_0^1 \frac{dt}{1-t^n(1-t)^{m-n}},$$

completing the proof.

Remark: As special cases of Theorem 2.1 we get

$$\sum_{k=0}^{\infty} \binom{2k}{k}^{-1} = \frac{4}{3} + \frac{2\pi\sqrt{3}}{27}, \tag{2}$$

$$\sum_{k=0}^{\infty} \binom{4k}{2k}^{-1} = \frac{16}{15} + \frac{\pi\sqrt{3}}{27} - \frac{2\sqrt{5}}{25} \ln \frac{1+\sqrt{5}}{2}. \tag{3}$$

Indeed, according to Theorem 2.1, we have

$$\sum_{k=0}^{\infty} \binom{2k}{k}^{-1} = \int_0^1 \frac{1+t-t^2}{(1-t+t^2)^2} dt = 2 \int_0^1 \frac{dt}{(1-t+t^2)^2} - \int_0^1 \frac{dt}{1-t+t^2} \tag{4}$$

and

$$\sum_{k=0}^{\infty} \binom{4k}{2k}^{-1} = \int_0^1 \frac{1+3t^2(1-t)^2}{(1-t^2(1-t)^2)^2} dt,$$

respectively. Since

$$\frac{1+3x^2}{(1-x^2)^2} = \frac{1+x}{2(1-x)^2} + \frac{1-x}{2(1+x)^2},$$

we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{4k}{2k}^{-1} &= \int_0^1 \frac{1+t-t^2}{2(1-t+t^2)^2} dt + \int_0^1 \frac{1-t+t^2}{2(1+t-t^2)^2} dt \\ &= \int_0^1 \frac{dt}{(1-t+t^2)^2} - \frac{1}{2} \int_0^1 \frac{dt}{1-t+t^2} + \int_0^1 \frac{dt}{(1+t-t^2)^2} - \frac{1}{2} \int_0^1 \frac{dt}{1+t-t^2}. \end{aligned} \tag{5}$$

Taking into account that

$$\int_0^1 \frac{dt}{1-t+t^2} = \frac{2\pi\sqrt{3}}{9} \quad \text{and} \quad \int_0^1 \frac{dt}{1+t-t^2} = \frac{4\sqrt{5}}{5} \ln \frac{1+\sqrt{5}}{2},$$

and that (see, e.g., [1])

$$\int \frac{dt}{(a+bt+ct^2)^2} = \frac{b+2ct}{(4ac-b^2)(a+bt+ct^2)} + \frac{2c}{4ac-b^2} \int \frac{dt}{a+bt+ct^2},$$

from (4) and (5) one can easily obtain (2) and (3).

Theorem 2.2 ([4], Theorem 2): If $n \geq 2$ is an integer, then

$$\sum_{k=0}^{\infty} \binom{n+k}{k}^{-1} = \frac{n}{n-1}.$$

Proof: For each positive integer p , we have

$$\begin{aligned} s_p &:= \sum_{k=0}^p \binom{n+k}{k}^{-1} = \sum_{k=0}^p (n+k+1) \int_0^1 t^k (1-t)^n dt \\ &= \int_0^1 \left\{ (n+1)(1-t)^n \sum_{k=0}^p t^k + (1-t)^n \sum_{k=0}^p kt^k \right\} dt \\ &= (n+1) \int_0^1 (1-t)^n dt - (n+1) \int_0^1 t^{p+1} (1-t)^{n-1} dt + \int_0^1 t(1-t)^{n-2} dt \\ &\quad - (p+1) \int_0^1 t^{p+1} (1-t)^{n-2} dt + p \int_0^1 t^{p+2} (1-t)^{n-2} dt. \end{aligned}$$

Formula (1) yields

$$s_p = \frac{n}{n-1} - (n-2)! \frac{(np+p+1)(p+1)!}{(p+n+1)!} - (n+1)(n-1)! \frac{(p+1)!}{(p+n+1)!}.$$

Taking into account that $n \geq 2$, we conclude that $s_p \rightarrow \frac{n}{n-1}$ when $p \rightarrow \infty$.

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