FIBONACCI AND LUCAS NUMBERS AS CUMULATIVE CONNECTION CONSTANTS[1]

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1. SEQUENCES OF CUMULATIVE CONNECTION CONSTANTS

Let us briefly introduce the notion of *cumulative connection constants*. For more details and related topics, the reader is referred to [2], [4], and [5].

Suppose two sequences $\{r_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ of complex numbers are given. Then one can introduce two associated sequences of polynomials $\{q_n(x)\}_{n\geq 0}$ and $\{p_n(x)\}_{n\geq 0}$ as follows:

- $q_0(x) = p_0(x) = 1$, and
- for any $n \geq 1$:
  $q_n(x) = q_{n-1}(x)\cdot(x-r_n)$,
  $p_n(x) = p_{n-1}(x)\cdot(x-s_n)$.

For any $n \geq 0$, the *connection constants* (or *generalized Lah numbers*) *relating the* (*root*) *sequence* $\{r_n\}_{n\geq 1}$ *to* $\{s_n\}_{n\geq 1}$ (or, equivalently, relating $\{q_n(x)\}_{n\geq 0}$ to $\{p_n(x)\}_{n\geq 0}$) are the complex numbers $L_{n,k}$ uniquely defined via the relationship

$$p_n(x) = \sum_{k=0}^{n} L_{n,k}\cdot q_k(x),$$

where we limit the sum to n since, clearly, $L_{n,k} = 0$ for any $k > n$. It is also easy to verify that $L_{n,n} = 1$ for any $n \geq 0$, our polynomials being monic. Moreover, we stipulate that $L_{n,k} = 0$ for negative values of k.

For any $n \geq 0$, the n^{th} *cumulative connection constant* (**ccc**, for short) is defined as

$$\mathscr{C}_n = \sum_{k=0}^{n} L_{n,k}.$$

We say that $\{\mathscr{C}_n\}_{n\geq 1}$ is the sequence of **ccc**'s *relating* $\{r_n\}_{n\geq 1}$ *to* $\{s_n\}_{n\geq 1}$. Notice that we stipulate not to start the sequence of **ccc**'s with $\mathscr{C}_0$ which always equals 1, as one may easily see.

The following examples provide very well-known sequences of **ccc**'s. For the sake of completeness, in the tables at the end of the paper we sketch the number sequences involved in these examples.

(i) Let $\{r_n\}_{n\geq 1} = 0, 0, 0, \ldots$, and $\{s_n\}_{n\geq 1} = -1, -1, -1, \ldots$. Here we have

$$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot x^k$$

which clearly yields $\mathscr{C}_n = 2^n$ (see Table 1).

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(ii) Let $\{r_n\}_{n\geq 1} = 0, 1, 2, \ldots$, and $\{s_n\}_{n\geq 1} = 0, 0, 0, \ldots$. Here we have

$$x^n = \sum_{k=0}^{n} S(n, k) \cdot (x)_k,$$

where $(x)_0 \equiv 1$, $(x)_k = \prod_{i=0}^{k-1}(x-i)$ for $k \geq 1$, are the *falling* (or *lower*) *factorials*, and $S(n, k)$ are the *Stirling numbers of the second kind*. Then $\mathscr{C}_n$ is the n^{th} *Bell number* $\mathscr{B}_n$ (see Table 2).

(iii) Conversely, let $\{r_n\}_{n\geq 1} = 0, 0, 0, \ldots$, and $\{s_n\}_{n\geq 1} = 0, 1, 2, \ldots$. Here we have

$$(x)_n = \sum_{k=0}^{n} s(n, k) \cdot x^k,$$

where $s(n, k)$ are the *Stirling numbers of the first kind*. Then $\mathscr{C}_1 = 1$ and $\mathscr{C}_n = 0$ for each $n \geq 2$ (see Table 3).

(iv) Let $\{r_n\}_{n\geq 1} = 0, 0, 0, \ldots$, and $\{s_n\}_{n\geq 1} = 0, -1, -2, \ldots$. Here we have

$$\langle x \rangle_n = \sum_{k=0}^{n} c(n, k) \cdot x^k,$$

where $\langle x \rangle_0 \equiv 1$, $\langle x \rangle_n = \prod_{i=0}^{n-1}(x+i)$ for $n \geq 1$, are the *rising* (or *upper*) *factorials*, and $c(n, k) = (-1)^{n-k} \cdot s(n, k)$ are the *signless Stirling numbers of the first kind*. Then $\mathscr{C}_n = n!$ (see Table 4).

(v) Let $\{r_n\}_{n\geq 1} = 1, q, q^2, \ldots$, and $\{s_n\}_{n\geq 1} = 0, 0, 0, \ldots$. Here we have

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q \cdot g_k(x),$$

where $g_0(x) \equiv 1$, $g_k(x) = \prod_{i=0}^{k-1}(x - q^i)$ for $k \geq 1$, are the *Gaussian polynomials*, and $\binom{n}{k}_q$ are the *Gaussian binomial coefficients*. In this case, $\mathscr{C}_n$ is the n^{th} *Galois number relative to* q, namely, $\mathscr{G}_{n,q}$, which is known to count the number of subspaces of an n-dimensional vector space over GF(q) (see, e.g., [1], Ch. II, Sec. 4). This example for $q = 2$ is sketched in Table 5.

These and other relevant examples may be found, e.g., in [1], [3], and [6]. In the sequel, we give instances of the notion of **ccc** that involve Fibonacci, Lucas, and other more general sequences.

2. CCC VERSUS FIBONACCI

We are now going to show that Fibonacci numbers can be seen as the sequence of **ccc**'s relating two specific integer sequences. A generalization of this statement is then provided in Proposition 2.3.

To prove our results, we need the following recurrence on the connection constants $L_{n,k}$ relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$.

Theorem 2.1 [5, Prop. 2]: For any n, k,

$$L_{n,k} = L_{n-1,k-1} + (r_{k+1} - s_n) \cdot L_{n-1,k}. \tag{1}$$

This theorem allows us to obtain a nice recurrence relation for the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of **ccc**'s relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$.

Proposition 2.1: For any $n \geq 1$,

$$\mathscr{C}_n = (1-s_n)\cdot\mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-1,k}\cdot r_{k+1}. \tag{2}$$

Proof: Just put "$\sum_{k=0}^{n}$" on both sides of recurrence (1) in Theorem 2.1. Then the claimed result follows by easy computation. □

We can now state our first result.

Proposition 2.2: Let $\{r_n\}_{n\geq 1}$ be the sequence $0, 0, 1, 0, 1, 0, 1, \ldots$, i.e., $r_1 - 0$ and, for any $k \geq 1$, $r_{2\cdot k} = 0$ and $r_{2\cdot k+1} = 1$. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence $0, 0, 0, \ldots$. Then the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of ccc's relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is the *Fibonacci sequence*.

Proof: By applying recurrence (1) to the connection constants relating our two sequences, we easily obtain $L_{1,0} = L_{2,0} = L_{2,1} = 0$. By recalling that $L_{n,n} = 1$ for any $n \geq 0$, and by the definition of ccc, we get

$$\mathscr{C}_1 = L_{1,0} + L_{1,1} = 1,$$
$$\mathscr{C}_2 = L_{2,0} + L_{2,1} + L_{2,2} = 1.$$

Let us compute $\mathscr{C}_n$ for $n \geq 3$. Since $\{s_n\}_{n\geq 1}$ is the null sequence, recurrence (2) becomes

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-1,k}\cdot r_{k+1},$$

where we can expand $L_{n-1,k}$ according to recurrence (1), and get

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} (L_{n-2,k-1} + r_{k+1}\cdot L_{n-2,k})\cdot r_{k+1}. \tag{3}$$

Now, note that our sequence $\{r_n\}_{n\geq 1}$ satisfies $r_n^2 = r_n$ for any $n \geq 1$. We can use this fact in (3) to obtain

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-2,k-1}\cdot r_{k+1} + \sum_{k=0}^{n-1} L_{n-2,k}\cdot r_{k+1}. \tag{4}$$

The first sum in (4) gives $L_{n-2,1} + L_{n-2,3} + L_{n-2,5} + \cdots + L_{n-2,n-2}\cdot r_n$, while the second expands to $L_{n-2,2} + L_{n-2,4} + L_{n-2,6} + \cdots + L_{n-2,n-2}\cdot r_{n-1}$. Moreover, $L_{n-2,0} = 0$, as one may easily verify by using recurrence (1). Therefore, (4) becomes

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-2} L_{n-2,k} = \mathscr{C}_{n-1} + \mathscr{C}_{n-2},$$

and our claim follows. □

Indeed, this proposition (as well as the others we shall prove) can also be seen in terms of sequences of polynomials. As stated in Section 1, the two root sequences $\{r_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ in Proposition 2.2 originate two sequences of polynomials. The former gives $\{\phi_n(x)\}_{n\geq 0}$ with $\phi_0(x) \equiv 1$, $\phi_1(x) \equiv x$, and $\phi_n(x) = \phi_{n-1}(x)\cdot x^{(n+1) \bmod 2}\cdot(x-1)^{n \bmod 2}$ for $n \geq 2$. The latter yields $\{x^n\}_{n\geq 0}$. Thus, for any $n \geq 1$, we have

$$x^n = \sum_{k=0}^{n} L_{n,k} \cdot \phi_k(x) \quad \text{and} \quad \mathcal{F}_n = \sum_{k=0}^{n} L_{n,k},$$

where $\mathcal{F}_n$ is the n^{th} Fibonacci number.

In Table 6 we summarize the sequences we have just coped with in Proposition 2.2. The result in Proposition 2.2 can be generalized as follows.

Proposition 2.3: For fixed integers $d \geq 1$ and $m \geq 1$, let $\{r_n\}_{n\geq 1}$ be the sequence

$$\underbrace{0, 0, \ldots, 0}_{d-1}, 1, \underbrace{0, 0, \ldots, 0}_{m-1}, 1, \underbrace{0, 0, \ldots, 0}_{m-1}, 1, \ldots,$$

i.e., $r_n = 1$ whenever $n = d + h \cdot m$ for $h \geq 0$, and $r_n = 0$ otherwise. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence. Then the sequence $\{\mathcal{C}_n\}_{n\geq 1}$ of **ccc**'s relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is

(a) $\mathcal{C}_1 = \mathcal{C}_2 = \cdots = \mathcal{C}_{d-1} = 1$,

(b) $\mathcal{C}_{d+i} = 2 + i$ for $0 \leq i \leq m-1$,

(c) $\mathcal{C}_n = \mathcal{C}_{n-1} + \mathcal{C}_{n-m}$ for $n \geq d+m$.

Proof: For (a), it is enough to verify via recurrence (1) that, for $0 \leq k \leq n \leq d-1$, we have $L_{n,k} = \delta_{n,k}$ (Kronecker's symbol). For (b), again recurrence (1) says that, for $0 \leq i \leq m-1$, we get $L_{d+i,k} = 0$ for $0 \leq k \leq d-2$, while $L_{d+i,k} = 1$ for $d-1 \leq k \leq d+i$.

Let us now turn to (c). For $n \geq d+m$, recurrence (2) is easily seen to be equivalent to

$$\mathcal{C}_n = \mathcal{C}_{n-1} + \sum_{h=0}^{\lfloor \frac{n-d}{m} \rfloor} L_{n-1,\, d+h\cdot m-1}. \tag{5}$$

By considering the structure of $\{r_n\}_{n\geq 1}$, and by repeatedly applying recurrence (1), we can write $L_{n-1,\, d+h\cdot m-1}$ in (5) as

$$L_{n-1,\, d+h\cdot m-1} = \sum_{j=1}^{m} L_{n-m,\, d+h\cdot m-j}.$$

We use this in (5) to obtain

$$\mathcal{C}_n = \mathcal{C}_{n-1} + \sum_{h=0}^{\lfloor \frac{n-d}{m} \rfloor} \sum_{j=1}^{m} L_{n-m,\, d+h\cdot m-j}. \tag{6}$$

It is easy to see that the double sum in (6) actually is $\sum_{k=d-m}^{n-m} L_{n-m,k}$. Furthermore, by recalling that $d \geq 1$, $m \geq 1$, and that we are considering the case $n \geq d+m$, it is also easy to see that $L_{n-m,k} = 0$ for $k \leq d-m-1$. In conclusion, we can write (6) as

$$\mathcal{C}_n = \mathcal{C}_{n-1} + \sum_{k=0}^{n-m} L_{n-m,k} = \mathcal{C}_{n-1} = \mathcal{C}_{n-m},$$

whence the result. □

Needless to remark, Proposition 2.2 is just a special case of Proposition 2.3, up to setting $d = 3$ and $m = 2$. More interesting is the case $d = m = 1$, which yields the constant sequence $\{r_n\}_{n\geq 1} = 1, 1, 1, \ldots$. By Proposition 2.3, the sequence $\{\mathcal{C}_n\}_{n\geq 1}$ of **ccc**'s relating such $\{r_n\}_{n\geq 1}$ to the null sequence is $\mathcal{C}_1 = 2$, $\mathcal{C}_n = 2 \cdot \mathcal{C}_{n-1}$, i.e., $\mathcal{C}_n = 2^n$. This is in perfect accordance with the well-known identity

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (x-1)^k .$$

As a further example, in Table 7 we display the case $d = m = 3$.

3. CCC VERSUS LUCAS

In this section we provide a further interesting generalization of the result in Proposition 2.2. As a simple consequence, we obtain two specific integer sequences whose associated sequence of ccc's is exactly the Lucas sequence.

Proposition 3.1: Given two complex numbers a and b, let $\{r_n\}_{n\geq 1}$ be the sequence defined as $r_1 = a$, $r_2 = b$, and $r_n = 1 - r_{n-1}^2$ for any $n \geq 3$. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence. Then the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of ccc's relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is

$$\mathscr{C}_n = \begin{cases} 1+a & \text{if } n = 1, \\ 1+a+a^2+b & \text{if } n = 2, \\ \mathscr{C}_{n-1}+\mathscr{C}_{n-2} & \text{if } n \geq 3. \end{cases}$$

Proof: The first two values of $\{\mathscr{C}_n\}_{n\geq 1}$ are derived at once by definition of ccc. Let us compute $\mathscr{C}_n$ for $n \geq 3$. Since $\{s_n\}_{n\geq 1}$ is the null sequence, recurrence (2) reads

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-1,k} \cdot r_{k+1} .$$

Now, we use recurrence (1) to expand $L_{n-1,k}$. We obtain

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} (L_{n-2,k-1} \cdot r_{k+1} + L_{n-2,k} \cdot r_{k+1}^2),$$

which, via the relation $r_n = 1 - r_{n-1}^2$, changes to

$$\mathscr{C}_n = \mathscr{C}_{n-1} + \sum_{k=0}^{n-1} L_{n-2,k-1} \cdot r_{k+1} + \sum_{k=0}^{n-1} L_{n-2,k} - \sum_{k=0}^{n-1} L_{n-2,k} \cdot r_{k+2} .$$

All terms of the first and the third sum cancel out, except the two terms $L_{n-2,-1} \cdot r_1$ and $L_{n-2,n-1} \cdot r_{n+1}$ that both equal 0, as noticed in Section 1. Since the second sum coincides with $\mathscr{C}_{n-2}$, our claim follows. □

It is easy to observe that Proposition 3.1 has Proposition 2.2 as a simple consequence, up to setting $a = b = 0$. Furthermore, it enables us to immediately get our claim on Lucas numbers as a sequence of ccc's.

Corollary 3.1: Let $\{r_n\}_{n\geq 1}$ be the sequence defined as in Proposition 3.1, up to setting $a = 0$ and $b = 2$. Moreover, let $\{s_n\}_{n\geq 1}$ be the null sequence. Then the sequence $\{\mathscr{C}_n\}_{n\geq 1}$ of ccc's relating $\{r_n\}_{n\geq 1}$ to $\{s_n\}_{n\geq 1}$ is the Lucas sequence.

In Table 8 we outline the sequences singled out in this corollary.

4. A FINAL REMARK

For the sake of precision, it is worth noticing that all sequences of **ccc**'s are meant to be determined *up to translation* of the related root sequences. More precisely: if $\{\mathscr{C}_n\}_{n\geq1}$ relates $\{r_n\}_{n\geq1}$ to $\{s_n\}_{n\geq1}$, then $\{\mathscr{C}_n\}_{n\geq1}$ also relates the translated sequences $\{r_n+\xi\}_{n\geq1}$ and $\{s_n+\xi\}_{n\geq1}$ for any complex number ξ. In fact, from Theorem 2.1, it is easy to verify that the connection constants relating $\{r_n\}_{n\geq1}$ to $\{s_n\}_{n\geq1}$ are the same that relate $\{r_n+\xi\}_{n\geq1}$ and $\{s_n+\xi\}_{n\geq1}$.

TABLE 1. Number of Subsets as Sequences of ccc's Arising from Binomial Coefficients $\binom{n}{k}$ (Ex. (i))

n	r_n	s_n	$\binom{n}{k}$								$\mathscr{C}_n=2^n$
			$k=0$	1	2	3	4	5	6	7	
1	0	−1	1	1							2
2	0	−1	1	2	1						4
3	0	−1	1	3	3	1					8
4	0	−1	1	4	6	4	1				16
5	0	−1	1	5	10	10	5	1			32
6	0	−1	1	6	15	20	15	6	1		64
7	0	−1	1	7	21	35	35	21	7	1	128

TABLE 2. Bell Numbers $\mathscr{B}_n$ as Sequences of ccc's Arising from Stirling Numbers of the Second Kind $S(n, k)$ (Ex. (ii))

n	r_n	s_n	$S(n,k)$								$\mathscr{C}_n=\mathscr{B}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	1	0	0	1	3						2
3	2	0	0	1	3	1					5
4	3	0	0	1	7	6	1				15
5	4	0	0	1	15	25	10	1			52
6	5	0	0	1	31	90	65	15	1		203
7	6	0	0	1	63	301	350	140	21	1	877

TABLE 3. The Sequence of ccc's Arising from Stirling Numbers of the First Kind $s(n, k)$ (Ex. (iii))

n	r_n	s_n	$s(n,k)$								$\mathscr{C}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	1	0	−1	1						0
3	0	2	0	2	−3	1					0
4	0	3	0	−6	11	−6	1				0
5	0	4	0	24	−50	35	−10	1			0
6	0	5	0	−120	274	−225	85	−15	1		0
7	0	6	0	720	−1764	1624	−735	175	−21	1	0

TABLE 4. Factorial Numbers as Sequences of ccc's Arising from Signless Stirling Numbers of the First Kind $c(n,k)=(-1)^{n-k}\cdot s(n,k)$ (Ex. (iv))

n	r_n	s_n	$c(n,k)$								$\mathscr{C}_n = n!$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	−1	0	1	1						2
3	0	−2	0	2	3	1					6
4	0	−3	0	6	11	6	1				24
5	0	−4	0	24	50	35	10	1			120
6	0	−5	0	120	274	225	85	15	1		720
7	0	−6	0	720	1764	1624	735	175	21	1	5040

TABLE 5. Galois Numbers $\mathscr{G}_{n,2}$ as Sequences of ccc's Arising from Gaussian Binomial Coefficients $\binom{n}{k}_2$ (Ex. (v))

n	r_n	s_n	$\binom{n}{k}_2$								$\mathscr{C}_n = \mathscr{G}_{n,2}$
			$k=0$	1	2	3	4	5	6	7	
1	1	0	1	1							2
2	2	0	1	3	1						5
3	4	0	1	7	7	1					16
4	8	0	1	15	35	15	1				67
5	16	0	1	31	155	155	31	1			374
6	32	0	1	63	651	1395	651	63	1		2825
7	64	0	1	127	2667	11811	11811	2667	127	1	29212

TABLE 6. Fibonacci Numbers $\mathscr{F}_n$ as Sequences of ccc's (Prop. 2.2)

n	r_n	s_n	$L_{n,k}$								$\mathscr{C}_n = \mathscr{F}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	0	0	0	1						1
3	1	0	0	0	1	1					2
4	0	0	0	0	1	1	1				3
5	1	0	0	0	1	1	2	1			5
6	0	0	0	0	1	1	3	2	1		8
7	1	0	0	0	1	1	4	3	3	1	13

TABLE 7. The Sequence of ccc's Arising from Prop. 2.3, for $d=m=3$

n	r_n	s_n	$L_{n,k}$								$\mathscr{C}_n$
			$k=0$	1	2	3	4	5	6	7	
1	0	0	0	1							1
2	0	0	0	0	1						1
3	1	0	0	0	1	1					2
4	0	0	0	0	1	1	1				3
5	0	0	0	0	1	1	1	1			4
6	1	0	0	0	1	1	1	2	1		6
7	0	0	0	0	1	1	1	3	2	1	9

TABLE 8. Lucas Numbers $\mathcal{L}_n$ as Sequences of ccc's (Cor. 3.1)

n	r_n	s_n	$L_{n,k}$ $k=0$	1	2	3	4	5	6	7	$\mathcal{C}_n = \mathcal{L}_n$
1	0	0	0	1							1
2	2	0	0	2	1						3
3	−3	0	0	4	−1	1					4
4	−8	0	0	8	7	−9	1				7
5	−63	0	0	16	−13	79	−72	1			11
6	−3968	0	0	32	55	−645	4615	−4040	1		18
7	−15745023	0	0	64	−133	5215	−291390	16035335	−15749063	1	29

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