MAXIMAL SUBSCRIPTS WITHIN GENERALIZED FIBONACCI SEQUENCES

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The generalized Fibonacci sequence $\{H_n\}$, where $H_n = H_{n-1} + H_{n-2} = H_n(a, b)$, $H_1 = a$, $H_2 = b$, a and b integers, has been studied in the classic paper by Horadam [8], and by Hoggatt [7], and Brousseau [1], [2], among others. In this paper we approach the problem of representing a positive integer N as a term in one of these generalized sequences so that, for $N = H_R(A, B)$, the subscript R is as large as possible.

Cohn [5] has solved a similar problem, but part of his theorem statement was omitted. Cohn's problem was: given a large positive integer N, find positive integers A, B such that the sequence $\{w_n\}$ defined by $w_1 = A$, $w_2 = B$, and $w_{n+2} = w_{n+1} + w_n$, $n \geq 1$, contains N, and $A + B$ is minimal.

Cohn's Theorem (Restated): Let $t_n = (-1)^n(NF_{n-1} - tF_n)$, where $t_k = A + B$, $t_{k+1} = B$, $t_{k+2} = A$. Then either $t = [N/\alpha]$ or $t = [N/\alpha] + 1$, where $[x]$ is the greatest integer in x and $\alpha = (1+\sqrt{5})/2$, gives the smallest value for $t_k = A + B$, depending upon n even or n odd.

Our problem has a different approach and allows computation of subscripts. The number $R_{\max}$ of this paper is related to a conjecture made by Hoggatt and proved by Klarner [9] that, for "n sufficiently large," $R(H_n - 1) = R(H_{n+1} - 1)$, where $R(N)$ is the number of representations of N as the sum of distinct Fibonacci numbers; $R_{\max}$ gives the value for n to be "sufficiently large" [3].

1. INTRODUCTION

In order to discuss maximal subscripts, we need a careful analysis of where we want the sequences $\{H_n(A, B)\}$ to begin. The Lucas sequence has $L_0 = 2$, with terms to the right strictly increasing, while $L_{-1} = -1$ is the first negative term in an alternating sequence to the left of L_0. A generalized Fibonacci sequence in which $H_{n+1} = H_n + H_{n-1}$, $H_1 = a \geq 1$, $H_2 = b \geq 1$, has $H_0 = b - a$, where we list terms to the right and the left of H_0 as

$$\ldots, 2b - 3a,\ 2a - b,\ H_0 = b - a,\ a,\ b,\ a + b,\ a + 2b, \ldots.$$

If we want $H_{-1} < 0$ as the first negative term, we need $b > 2a$; then $(2a - b) < 0$ as well as $(b - a) > a$ and $b > a$. Then, $H_1 = a$ is the smallest positive term in the generalized sequence and the terms to the right of H_0 are strictly increasing. The Fibonacci sequence, however, has $F_0 = 0$ with strictly increasing terms to the right of $F_1 = 1$, and the sequences $\{aF_n\}$ are the only sequences $\{H_n\}$ which contain $H_k = 0$. We write $H_{-1} = 0$, $H_0 = a \geq 1$, $H_1 = a$, $H_2 = b = 2a$:

$$\ldots, -3a,\ 2a,\ -a,\ a,\ 0,\ H_0 = a,\ a,\ 2a,\ 3a,\ 5a, \ldots.$$

Notice that the sequence $H_n = aF_{n+1}$ has the same characteristics as the Lucas-like sequence $H_1 = a \geq 1$, $H_2 > 2a$: $H_{-1} = 0$ is the first nonpositive term in an alternating sequence moving left of $H_0 = a$, while terms to the right of H_0 are strictly increasing. $H_1 = H_0 = a$ are the smallest positive terms.

Thus, we define the *standardized generalized Fibonacci (S.G.F.) sequence* $\{H_n(a, B)\}$ by

$$H_{n+1} = H_n + H_{n-1},\ H_1 = A \geq 1,\ H_2 = B \geq 2A.$$

We note that $H_0 = B - A \geq A$, and $H_1 = A$ is the smallest term in the sequence. We will find that H_0 will determine maximal subscripts for the sequence. If $B = 2A$, we will have a Fibonacci-like sequence in which $H_n = AF_{n+1}$. Also, Fibonacci and Lucas numbers are numbered to be consistent with usage in this journal.

We need this careful definition of the beginning terms so that we can identify $H_1 = A$ and $H_2 = B$ given any two adjacent terms somewhere in the sequence. For example, 13 and 17 are adjacent in each of $\{13, 17, 30, 47, \ldots\}$, $\{4, 13, 17, 30, \ldots\}$, and $\{9, 4, 13, 17, 30, \ldots\}$. Note that the S.G.F. sequence will have $A = 4$, $B = 13$. We do not start with $A = 9$, $B = 4$, or with $A = 13$, $B = 17$, since a S.G.F. sequence must have $B \geq 2A$. While $N = H_1$ in an infinite number of such sequences, $N = H_n$, $n \geq 2$, can appear only within a S.G.F. sequence for which $1 \leq H_{n-1} \leq N - 1$. When $N = H_2$, take $1 \leq H_1 \leq [(N-1)/2]$, while $N = H_n$, $n \geq 3$, has $[N/2] + 1 \leq H_{n-1} \leq N - 1$. Thus, the maximal subscript for N can be found by listing possibilities. If $N = 7 = H_n$, examine sequences for which $4 \leq H_{n-1} \leq 6$, giving $1, 3, 4, 7, 11, \ldots$; $2, 5, 7, 12, \ldots$; and $1, 6, 7, 13, \ldots$. The first sequence has $7 = H_4$, and 4 is the maximal subscript for 7. If $N = 6 = H_n$, examine $4 \leq H_{n-1} \leq 5$: $2, 4, 6, 10, \ldots$, and $1, 5, 6, 11, \ldots$. Both sequences have $6 = H_3$, but the first sequence has $B = 2A$ so that $H_3 = 2F_4$; we take the larger subscript, and 4 is the maximal subscript for 6.

Lemma 1.1 gives a second way to compute maximal subscripts.

Lemma 1.1: If $H_n = H_{n-2} + H_{n-1}$, $H_1 = a$, $H_2 = b$, the equation

$$N = H_n(a, b) = aF_{n-2} + bF_{n-1} \tag{1.1}$$

has a solution for any integer N. If (a_0, b_0) is a solution for (1.1), then $a = a_0 - tF_{n-1}, b = b_0 + tF_{n-2}$ is also a solution for (1.1) for any integer t.

Proof: Equation (1.1), which can be proved by mathematical induction, always has solutions [10] for integers a, b as above since $(F_{n-2}, F_{n-1}) = 1$. □

For our purposes in using (1.1), $\{H_n(a, b)\}$ must be a S.G.F. sequence. Note that

$$\{H_n(1, 2)\} = \{F_{n+1}\} \quad \text{and} \quad \{H_n(1, 3)\} = \{L_n\}$$

are S.G.F. sequences since $B \geq 2A$, but while $\{H_n(1, 1\} = \{F_n\}$, this is not a S.G.F. sequence. If $F_{n-1} < N < F_n$, then $(n-2)$ is the largest possible subscript for N in a S.G.F. sequence by examining (1.1). If $N = 31$, since $F_8 < 31 < F_9$, solve $31 = H_7 = AF_5 + BF_6$. We find $31 = H_7(3, 2)$ but $B < 2A$, so we solve $31 = AF_4 + BF_5 = H_6(2, 5)$, where $B > 2A$, obtaining 6 as the maximal subscript for 31. We now have two methods to compute a table of maximal subscripts.

We will say that *a natural number N reaches maximum expansion at R, denoted by* $\rho(N) = R$, if R is the largest subscript possible for N as a member of a S.G.F. sequence or for N as a member of a Fibonacci-like sequence. Let R be the largest subscript such that

$$N = H_R(A, B) = AF_{R-2} + BF_{R-1}$$

for $1 \leq A$ and $2A \leq B$. Then, if $2A < B$, $\rho(N) = R$; if $2A = B$, $\rho(N) = R+1$. We will find $\rho(F_R) = R = \rho(L_R)$ for $R \geq 3$. For the reader's convenience, we list maximal subscripts $\rho(N)$ in Table 1.

TABLE 1. $N = H_R(A, B)$ with Maximal $R = \rho(N)$

N	R	N	R	N	R	N	R
1	2	26	7	51	6	76	9
2	3	27	5	52	7	77	7
3	4	28	6	53	7	78	7
4	3	29	7	54	6	79	7
5	5	30	5	55	10	80	6
6	4	31	6	56	6	81	8
7	4	32	6	57	6	82	7
8	6	33	6	58	7	83	6
9	4	34	9	59	6	84	8
10	5	35	5	60	8	85	7
11	5	36	6	61	7	86	8
12	4	37	7	62	6	87	7
13	7	38	6	63	8	88	6
14	5	39	7	64	6	89	11
15	5	40	6	65	7	90	7
16	6	41	6	66	7	91	7
17	5	42	8	67	6	92	7
18	6	43	6	68	9	93	7
19	5	44	6	69	7	94	8
20	5	45	7	70	6	95	7
21	8	46	6	71	7	96	6
22	5	47	8	72	6	97	9
23	6	48	6	73	8	98	7
24	6	49	6	74	7	99	8
25	5	50	7	75	6	100	7

2. $\rho(N)$ FOR SOME SPECIAL INTEGERS N

We write $\rho(N)$ for some specialized integers N and consider how many integers N have $R = \rho(N)$ for a given subscript R. If $\rho(N) = R$ in exactly one sequence, N is called a *single*; if in exactly two sequences, N is called a *double*; if in exactly three sequences, N is called a *triple*. The smallest double occurs when $R = 3$, for $N = 4 = H_3(1, 3) = 2F_3$, while the smallest triple occurs when $R = 5$, for $N = 35 = 7F_5 = H_5(4, 9) = H_5(1, 11)$.

Theorem 2.1: For the Fibonacci sequence, $\rho(AF_R) = R$ when $1 \leq A < F_{R+1}$, $R \geq 2$. Further, $\rho(AF_R) > R$ when $A > F_{R+1}$.

Proof: $\rho(F_2) = 2$; $\rho(F_3) = 3$. By Lemma 1.1,

$$AF_R = AF_{R-2} + AF_{R-1} = H_R(A + F_{R-1}, A - F_{R-2}),$$

where $A + F_{R-2} > 2(A - F_{R-1})$ when $A < F_{R+1}$ and $\rho(AF_R) \geq R$. Further, $AF_R = 0F_{R-1} + AF_R = H_{R+1}(F_R, A - F_R)$, but a S.G.F. sequence requires that $B > 2A$, and $A - F_{R-1} > 2F_R$ only when $A > F_{R+1}$. Thus, $\rho(AF_R) < R+1$, making $\rho(AF_R) = R$ when $A < F_{R+1}$, and $\rho(AF_R) > R$ when $A > F_{R+1}$. □

Corollary 2.1.1: Let $N = AF_R$, and $\rho(N) = R$, $R \geq 3$. Then N is a single when $A \leq F_{R-1}$; is a double when $F_{R-1} < A \leq 2F_{R-1}$; and is a triple when $2F_{R-1} < A < F_{R+1}$. For each value of R, $\rho(N) = R$ in at most three sequences.

Proof: If $1 \leq A < F_{R+1}$, $\rho(AF_R) = R$ by Theorem 2.1. By Lemma 1.1, any other solutions for $\rho(N) = R$ are found from $N = H_R(A - tF_{R-1}, A + tF_{R-2})$. If $t \leq 0$, $\{H_R\}$ is not a S.G.F. sequence. If $t = 1$, then S.G.F. sequence requirements dictate $A - F_{R-1} \geq 1$, making N a single when $1 \leq A \leq F_{R-1}$, and at least a double if $A > F_{R-1}$. If $t = 2$, we must have $A - 2F_{R-1} \geq 1$, so that N is a double when $F_{R-1} < A \leq 2F_{R-1}$, and a triple when $2F_{R-1} < A < F_{R+1}$. If $t \geq 3$, then $A \geq 1 + tF_{R-1} \geq 1 + F_{R+1}$, and $\rho(N) > R$. □

Corollary 2.1.2: For $R \geq 2$, $\rho(F_R^2) = R$ and $\rho(F_{R+1}F_R) = R+1$; further, $\rho(F_R^2 - 1) = R+1$, R even, and $\rho(F_R^2 + 1) = R+1$, R odd.

Proof: Apply Theorem 2.1 to $F_{R-1}F_{R+1} = F_R^2 + (-1)^R$. □

Corollary 2.1.3: $\rho(F_R L_{R-1}) = R$, $R \geq 2$; $\rho(F_R L_R) = 2R$, $R \geq 1$.

Proof: Since $L_{R-1} = F_R + F_{R-2} < F_{R+1}$, Theorem 2.1 gives $\rho(F_R L_{R-1}) = R$ and $\rho(F_R L_R) = \rho(F_{2R}) = 2R$. □

Corollary 2.1.4: $\rho(L_{n+1}F_n) = n+2$, $n \geq 3$; and $\rho(L_{n+k}F_k) = n+k$, $k \geq 2$, $n \geq 1$.

Proof: Let $N = L_{n+1}F_n = (F_{n-2})F_n + (2F_n)F_{n+1} = H_{n+2}(F_{n-2}, 2F_n)$; as $B > 2A$, $\rho(N) \geq n+2$. Since $N = H_{n+3}(F_{n+1}, F_{n-2})$ has no other positive solutions and $\{H_{n+3}(F_{n+1}, F_{n-2})\}$ is not a S.G.F. sequence, we have $\rho(N) < n+3$, making $\rho(N) < n+2$. Next, let $N = L_{n+k}F_k$. One can derive

$$N = L_{n+k}F_k = (F_k)F_{n+k-2} + (3F_k)F_{n+k-1} = H_{n+k}(F_k, 3F_k),$$

where $B > 2a$ and $\rho(N) \geq n+k$. Also, since $N = H_{n+k+1}(2F_k, F_k)$ has no other positive solutions for A and B, and this solution cannot be used because $A > B$, we have $\rho(N) < n+k+1$; thus, $\rho(L_{n+k}F_k) = n+k$. □

Corollary 2.1.5: $\rho(F_{n+p} + F_{n-p}) = n = \rho(F_{n+p} - F_{n-p})$, $p \geq 2$, $n \geq 2+p$

Proof: Hoggatt (see [7], p. 59] gives

$$F_{n+p} + F_{n-p} = F_n L_p,\ p \text{ even};\quad F_{n+p} + F_{n-p} = L_n F_p,\ p \text{ odd}.$$

If p is even, Corollary 2.1.3 gives $\rho(F_n L_p) = n$; if p is odd, Corollary 2.1.4 gives $\rho(L_n F_p) = n$. Similarly, $F_{n+p} - F_{n-p} = F_n L_p$, p odd, and $F_{n+p} - F_{n-p} = L_n F_p$, p even, yield $\rho(F_{n+p} - F_{n-p}) = n$. □

Corollary 2.1.6: $\rho(F_{2k} - 1) = \rho(F_{2k} + 1) = k+1$, $k \geq 2$.

If k is even, $k \geq 4$, $\rho(F_{2k+1} + 1) = \rho(F_{2k} + 1) = k+1$;
$\rho(F_{2k+1} - 1) = \rho(F_{2k} - 1) + 1 = k+2$.

If k is odd, $k \geq 3$, $\rho(F_{2k+1} - 1) = \rho(F_{2k} - 1) = k+1$;
$\rho(F_{2k+1} + 1) = \rho(F_{2k} + 1) + 1 = k+2$.

Proof: When k is odd, $k \ge 1$:

$$F_{2k}+1 = F_{k+1}L_{k-1}; \qquad F_{2k}-1 = F_{k-1}L_{k+1};$$
$$F_{2k+1}-1 = F_{k+1}L_k. \qquad F_{2k+1}+1 = F_kL_{k+1}.$$

When k is even, $k \ge 2$:

$$F_{2k}+1 = F_{k-1}L_{k+1}; \qquad F_{2k}-1 = F_{k+1}L_{k-1};$$
$$F_{2k+1}-1 = F_kL_{k+1}. \qquad F_{2k+1}+1 = F_{k+1}L_k.$$

Each pair of identities, when summed vertically, gives $F_{2k+2} = F_{k+1}L_{k+1}$, and each can be proved by mathematical induction. Then apply Corollaries 2.1.3 and 2.1.4, which give $\rho(N)$ when $N = F_kL_m$. □

Next, we investigate integers N, where $\rho(N) = R$ and N is not a multiple of F_R. The smallest such double is $N = 83 = H_6(1, 16) = H_6(6, 13)$.

Theorem 2.2: Let $N = H_R(A, B)$, where $B > 2A$ and $\rho(N) = R$, $R \ge 3$. Then N is a single when $1 \le A \le F_{R-1}$ and $B < A + F_R$. N is a double when $F_{R-1} < A \le F_R - 2$ and $2A < B \le 2F_R - 3$.

Proof: Select the smallest integer A for which the hypothesis is met. Then $1 \le A \le F_{R-1}$. Otherwise, from Lemma 1.1, $N = H_R(A - F_{R-1},\ B + F_{R-2})$, contrary to choice of A as smallest. $\{H_R(A + F_{R-1},\ B - F_{R-2})\}$ is not a S.G.F. sequence when $B < A + F_R$ because then $A + F_{R-1} > B - F_{R-2}$; thus, the conditions $A \le F_{R-1}$ and $B < A + F_R$ guarantee a single. When $A \ge 1 + F_{R-1}$ and $B < A + F_R$, $\{H_R(A - F_{R-1},\ B + F_{R-2})\}$ is a S.G.F. sequence. Since $2A < B$, rewrite requirements for B as $2A + 1 \le B \le F_R + A + 1$, leading to a double when $F_{R-1} + 1 \le A \le F_R - 2$ and $2A + 1 \le B \le F_R + A + 1 \le 2F_R - 3$. □

To illustrate Theorem 2.2, consider $H_6(1, 16) = H_6(6, 13) = 83$. The smallest solution has $A = 1$, where $1 \le A \le F_{6-1}$, but $B = 16 > F_6 + A = 9$, allowing a double. Taking $A = 6$, $F_5 < 6 \le F_6 - 2$ and $B = 13 \le 2F_6 - 3$.

Corollary 2.2.1: Let $N = H_R(A, B)$, where $B > 2A$. If $1 \le A \le F_R - 2$ and $B < A + F_R$, then $\rho(N) = R$, $R \ge 3$.

Proof: By hypothesis, $\rho(N) \ge R$.

$$N = AF_{R-2} + BF_{R-1} = (B - A)F_{R-1} + AF_R = H_{R+1}(B - A, A),$$

but $\{H_{R+1}(B - A, A)\}$ is not a S.G.F. sequence when $B > 2A$, and neither are the other solutions from Lemma 1.1, $N = H_{R+1}(B - A + F_R, A - F_{R-1})$ and $N = H_{R+1}(B - A - F_R, A + F_{R-1})$. Thus, $\rho(N) < R + 1$ and $\rho(N) = R$. □

Corollary 2.2.2: $\rho(N) = R$ for $(F_R^2 + F_{R-3})/2$ integers N, $R \ge 3$.

Proof: When $B > 2A$, Corollary 2.2.1 gives $(F_R - 2)$ choices for A. Since $2A + 1 \le B \le F_R + A + 1 \le 2F_R - 3$, when $A = F_R - 2$, there is one choice for B; for $A = F_R - 3$, two choices; ..., for $A = F_R - 1 - k$, k choices. So $\rho(N) = R$ for

$$(F_R - 2)(1 + 2 + 3 + \cdots + (F_R - 2)) = (F_R - 2)(F_R - 1)/2$$

integers N which are not divisible by F_R. If $N = AF_R$, Theorem 2.1 gives $\rho(N) = R$ for $1 \le A \le F_{R+1} - 1$, so there are $(F_{R+1} - 1)$ such integers N. Adding and simplifying,

$$(F_R - 2)(F_R - 1)/2 + (F_{R+1} - 1) = (F_R^2 + F_{R-3})/2$$

as required. □

Theorem 2.3: For the Lucas sequence, $\rho(L_R) = R$, $R \geq 3$.

Proof: $\rho(L_1) = 2$ and $\rho(L_2) = 4$. For $R \geq 3$, $L_R = H_R(1,3) = 1F_{R-2} + 3F_{R-1}$, so $\rho(L_R) \geq R$. The only positive solution for $L_R = H_{R+1}(A, B) = AF_{R-1} + BF_R$ is $A = 2$ and $B = 1$, but this solution cannot be used since $A > B$, so $\rho(L_R) < R+1$, making $\rho(L_R) = R$. Compare with Corollary 2.1.4 for $k = 2$. □

Corollary 2.3.1: The smallest integer such that $\rho(N) = R$ is F_R. The smallest integer not divisible by F_R such that $\rho(N) = R$ is L_R.

Theorem 2.4: The largest integer N for which $\rho(N) = R$ is $N = (F_{R+1} - 1)F_R$, $R \geq 2$. Also, $N = (F_{R+1} - 1)F_R$, $R \geq 5$, is a triple, with the other two occurrences given by

$$N = H_R(F_R - 1), 2F_R - 1) = H_R(F_{R-2} - 1, 2F_R + F_{R-2} - 1).$$

Proof: For $R = 2$, $N = (F_3 - 1)F_2 = 1$. If $H_R = AF_{R-2} + BF_{R-1}$, where $B \geq 2A$, then $H_R \leq BF_{R-2} + BF_{R-1} = BF_R$, $R \geq 3$. $\rho(BF_R) = R$ when $1 \leq B \leq F_{R+1} - 1$ by Theorem 2.1. By Corollary 2.1.1, $N = (F_{R+1} - 1)F_R$ is a triple that can be calculated using Lemma 1.1. □

Theorem 2.5: If $F_{2k-2} \leq N < F_{2k}$, $k \geq 2$, then $k \leq \rho(N) \leq 2k - 1$.

Proof: By Theorem 2.1, the largest possible value for $\rho(N)$ in the interval is $\rho(F_{2k-1}) = 2k - 1$. We show that the smallest value is $\rho(N) = k$ by applying Theorem 2.4. Now, take $N = (F_{k+1} - 1)F_k$; then $N < (F_{k+1} + F_{k-1})F_k = L_kF_k = F_{2k}$, while

$$N = (F_k + (F_{k-1} - 1))F_k \geq (F_k + F_{k-2})F_{k-1} = L_{k-1}F_{k-1} = F_{2k-2}$$

for $k \geq 4$. Then, by examining $k = 2$ and $k = 3$, and putting this together,

$$F_{2k-2} \leq N = (F_{k+1} - 1)F_k < F_{2k}, \; k \geq 2, \tag{2.5.1}$$

where $\rho(N) = k$ and N is the largest integer such that $\rho(N) = k$. Notice that taking $R = k - 1$ in Theorem 2.4, $(F_k - 1)F_{k-1} < F_{2(k-1)}$ from (2.5.1), so that the largest integer N having $\rho(N) = k - 1$ is not in the interval of Theorem 2.5. □

Theorem 2.6: In the interval $F_m < N < F_{m+1}$, $[(m+2)/2] \leq \rho(N) \leq m - 1$, $m \geq 4$; and $\rho(F_m + 1) \leq [(m+2)/2] + 1$, where $[x]$ is the greatest integer in x.

Proof: Since F_m is not in the interval, $\rho(N) \leq m - 1$. If m is odd, take $m = 2k - 1$; if m is even, $m = 2k - 2$. Either $F_{2k-2} \leq N < F_{2k-1}$ or $F_{2k-1} \leq N < F_{2k}$, so that $\rho(N) \geq k$ from Theorem 2.5. Since either $[(m+2)/2] = (2k - 1 + 2)/2] = k$ or $[(m+2)/2] = (2k - 2 + 2)/2] = k$, $\rho(N) \geq [(m+2)/2]$.

The smallest integer in the interval is $F_m + 1$, and, by Corollary 2.1.6, either $\rho(F_m + 1) = [(m+2)/2]$ or $\rho(F_m + 1) = [(m+2)/2] + 1$. The largest value for $\rho(N)$ is $m - 1$, which occurs for $N = 2F_{m-1}$ and $N = L_{m-1}$. □

Corollary 2.6.1: For $m \geq 4$,

$$\rho(F_m + F_{m-2}) = \rho(F_{m+1} - F_{m-2}) = m - 1;$$
$$\rho(F_m + F_{m-3}) = \rho(F_{m+1} - F_{m-3}) = m - 1.$$

Proof: Since $L_{m-1} = F_m + F_{m-2} = F_{m+1} - F_{m-3}$, apply Theorem 2.3 in the first case. Similarly, use Theorem 2.1 with $2F_{m-1} = F_{m+1} - F_{m-2} = F_m + F_{m-3}$. □

3. THE MAXIMUM EXPANSION INDEX OF A S.G.F. SEQUENCE

In this section, we determine when $\rho(H_K) = K$ for the S.G.F. sequence $\{H_n(A, B)\}$. We will call the integer $R_{\max}$ the *maximum expansion index* of the S.G.F. sequence $\{H_n(A, B)\}$ if $\rho(H_K(A, B)) = K$ whenever $K = R_{\max}$. For example, the S.G.F. sequence

$$\{H_n(1, 7)\} = \{1, 7, 8, 15, 23, 38, 61, 99, \ldots\}$$

which has $\rho(H_6) = \rho(38) = 6$ has $R_{\max} = 6$; $\rho(H_K) \neq K$ for $1 \leq K \leq 5$, while $\rho(H_7) = \rho(61) = 7$ as well as $\rho(H_K) = K$ for all $K \geq 6$.

Theorem 3.1: If $F_{R-1} < B - A \leq F_R$ for the S.G.F. sequence $\{H_n(A, B)\}$, then $\rho(H_R(A, B)) = R$. Further, $R = R_{\max}$, and $\rho(H_K(A, B)) = K$ for all $K \geq R$.

Proof: Since $2A \leq B$ in a S.G.F. sequence, $A \leq B - A \leq F_R$ so $1 \leq A \leq F_R$ and $B \leq A + F_R$.. If $B = 2A$, then $N = AF_R$ and $\rho(N) = R$ by Theorem 2.1. Also, $B = A + F_R$ gives a Fibonacci-like case, since $A = F_R - k$, $B = 2F_R - k$ give

$$N = H_R = (F_R - k)F_{r-2} + (2F_R - k)F_{r-1} = (F_{R+1} - k)F_R, \ 1 \leq k \leq F_R - 1,$$

where $\rho(N) - R$ by Theorem 2.1, and $k = 0$ gives us $B = 2A$, already discussed.

If $B > 2A$, Corollary 2.2.1 gives $\rho(N) = R$ when $1 \leq A \leq F_R - 2$, $B < A + F_R$, leaving only the cases $A = F_R - 1$ and $A = F_R$. Since cases $B = 2A$ and $B = A + F_R$ were discussed above, we are finished, and $\rho(N) = R$ when $F_{R-1} < B - A \leq F_R$.

Let $K > R$. If

$$N = H_K(A, B) = AF_{K-2} + BF_{K-1} \quad \text{and} \quad B \geq 2A,$$

then $\rho(N) \geq K$. Thus,

$$N = H_{K+1}(B - A, A) = (B - A)F_{K-1} + AF_K$$

but this solution cannot be used since $B - A \geq A$ when $B \geq 2A$. Since $F_K > F_R$, $(B - A) - F_K < 0$, and $(B - A) + F_K > A - F_{K-1}$ when $B \geq 2A$, Lemma 1.1 gives no other usable solutions for $N = H_{k+1}$. Thus, $\rho(N) < K + 1$ and $\rho(N) = K$. Putting these cases together, $\rho(H_K(A, B)) = K$ when $K \geq R$, and $R = R_{\max}$. □

Corollary 3.1.1: If $F_{R-1} < 2a \leq F_R$, $a \geq 1$, $R \geq 3$, then $\rho(aL_n) = n$ for $n \geq R$. If $F_{R-1} < A < F_R$, then $\rho(AF_n) = n$ for $n \geq R$, $R \geq 2$.

Proof: $H_n = aL_n$ has $H_0 = 2a$. If $F_{R-1} < B - A = 2a \leq F_R$, apply Theorem 3.1. If $F_{R-1} < 2A - A \leq F_R$, then $\rho(H_{n-1}(A, 2A)) = n - 1$ for $n - 1 \geq R$. Since $B = 2A$ and $AF_n = H_{n-1}(A, 2A)$, $\rho(AF_n) = n$ for $n \geq R$. Compare with Theorem 2.1. □

Theorem 3.2: For $k \geq 2$, $n \geq 2$,

$$\rho(F_{n+2k}+F_n)=\rho(F_{n+2k}-F_n)=n+k;$$
$$\rho(F_{n+2k+1}+F_n)=\rho(F_{n+2k+1}-F_n)+1=n+k+1,\ k\text{ even};$$
$$\rho(F_{n+2k+1}+F_n)=\rho(F_{n+2k+1}-F_n)-1=n+k,\ k\text{ odd}.$$

Proof: $\rho(F_{n+2k}+F_n)=\rho(F_{n+2k}-F_n)=n+k$ by taking $n=n+k$ and $p=k$ in Corollary 2.1.5. Since $N=H_{n+k}=AF_{n+k-2}+BF_{n+k-1}$, where $\rho(H_{n+k})\geq n+k$ if $B\geq 2A$, we derive identities involving F_{n+2k+1} from the identity (see Eq. (8) in [11])

$$F_{m+n}=F_{m-1}F_n+F_mF_{n+1} \tag{2.7.1}$$

to write $N=F_n+F_{n+2k+1}=AF_{n+k-1}+BF_{n+k}$. Take $m=n+k$ and $n=k+1$ for F_{n+2k+1} and $m=n+k$, and $n=(-k)$ for F_n in (2.7.1) to write

$$F_{n+2k+1}=F_{(n+k)+(k+1)}=F_{n+k-1}F_{k+1}+F_{n+k}F_{k+2};$$
$$F_n=F_{(n+k)+(-k)}=F_{n+k-1}F_{-k}+F_{n+k}F_{1-k}=(-1)^{k+1}F_kF_{n+k-1}+(-1)^kF_{k-1}F_{n+k}.$$

Thus,

$$N=F_{n+2k+1}+F_n=(F_{k+1}+(-1)^{k+1}F_k)F_{n+k-1}+(F_{k+2}+(-1)^kF_{k-1})F_{n+k}$$
$$=H_{n+k+1}(A,B).$$

If k is even, $A=F_{k+1}-F_k=F_{k-1}$, and $B=F_{k+2}+F_{k-1}=2F_{k+1}$, where $B>2A$. Since $B-A=F_{k+2}$, $R_{\max}=k+2$, where $n+k+1\geq k+2$; $\rho(N)=n+k+1$ by Theorem 3.1. If k is odd, then $A=F_{k+1}+F_k=F_{k+2}$ and $B=F_{k+2}-F_{k-1}=2F_k$ has $A>B$ with no other positive solution, but we find that $N=H_{n+k}(2F_k,4F_k+F_{k-1})$, where $F_{k+1}<B-A\leq F_{k+2}$ so that $R_{\max}=k+2$, and again $\rho(N)=n+k, n\geq 2$..

Subtracting the quantities above, $N=F_{n+2k+1}-F_n$ becomes $H_{n+k}(2F_k,4F_k+F_{k-1})$, giving $\rho(N)=n+k$ for k even; for k odd, $N=F_{n+2k+1}-F_n$ becomes $H_{n+k+1}(F_{k-1},2F_{k+1})$, giving $\rho(N)=n+k+1$. □

4. SOLVING $N=H_R(A,B)$ FOR R, A, AND B

Given N, we find A, B, and R so that $N=H_R(A,B)$, where $R=\rho(N)$. Our solution depends upon a greatest integer identity for the S.G.F. sequence $\{H_n(A,B)\}$ which allows us to find H_{n-1} when we are given H_n.

Lemma 4.1: Let $\{H_n(A,B)\}$ be a S.G.F. sequence, where $F_{k-1}<B-A\leq F_k$. For $n\leq k$, the term preceding $H_n(A,B)$ is $[H_n/\alpha]$ or $[H_n/\alpha]+1$, where $[x]$ is the greatest integer in x, and $\alpha=(1+\sqrt{5})/2$.

Proof: From [4], use Theorem 3.3 when $B>2A$ and Theorem 2.3 when $B=2A$. □

Lemma 4.2: For the S.G.F. sequence $\{H_n(A,B)\}$, if $D=B^2-AB-A^2$:

(i) $F_{n-1}<H_n/\sqrt{D}\leq F_{n+1}$;

(ii) $H_n^2-H_nH_{n-1}-H_{n-1}^2=(-1)^nD$;

(iii) $|H_n^2-H_nH_{n-1}-H_{n-1}^2|=K^2$ iff $H_n=KF_{n+1}$, $n\geq 1$.

Proof: (1) Since $B \ge 2A$, $D > 0$; in fact, $D \ge B^2/4$, and $\sqrt{D} \ge B/2$. Thus,

$$H_n = AF_{n-2} + BF_{n-1} \le (B/2)F_{n-2} + BF_{n-1} = (B/2)F_{n+1}.$$

Dividing by $\sqrt{D}$, $H_n/\sqrt{D} \le (B/2)F_{n+1}/\sqrt{D} \le F_{n+1}$, while

$$F_{n+1} \ge H_n/\sqrt{D} = (AF_{n-2} + BF_{n-1})/\sqrt{D} > AF_{n-2}/\sqrt{D} + F_{n-1} > F_{n-1}.$$

For (ii), see [1], [7], and [8]. Lastly, in 1876, Lucas proved that $m^2 - mn - n^2 = \pm 1$ is satisfied by consecutive Fibonacci numbers, and in 1902, Wasteels proved that there are no other solutions (see [6], p. 405). Since $(F_n, F_{n+1}) = 1$, (iii) follows. Note that (iii) is a test for a Fibonacci sequence. □

Lemma 4.3: Let $N = H_n(A, B)$, where n is to be maximized. There are two cases:

(i) $H_{n-1} = [H_n/\alpha]$, $n = n_1$;

(ii) $H_{n-1} = [H_n/\alpha] + 1$, $n = n_2$.

The maximal subscript value for $N = H_R$ occurs for $R = \max(n_1, n_2)$.

Proof: Lemma 4.3 actually is a blueprint for solving for R. By Lemma 4.1, cases (i) and (ii) give the only two possible choices for H_{n-1}. Take case (i). Compute $H_n^2 - H_nH_{n-1} - H_{n-1}^2 = (-1)^n D$ from Lemma 4.2 recalling that $D > 0$. Compute $H_n/\sqrt{D}$ and select n by $F_{n-1} < H_n/\sqrt{D} \le F_{n+1}$. There are two possibilities for n: if $(-1)^n D > 0$, then n is the even possibility, while n is odd if $(-1)^n D < 0$. Then $n = n_1$ is the solution from case (i). Now take case (ii). Make the same calculations with $H_{n-1} = [H_n/\alpha] + 1$ to find $n = n_2$. Then choose $n = R = \max(n_1, n_2)$. □

Lemma 4.4: If $N = H_n(A, B)$, then

$$A = |H_{n-1}F_{n-1} - NF_{n-2}| \quad \text{and} \quad B = |H_{n-1}F_{n-2} - NF_{n-3}|.$$

Proof: Refer to (1.1) and solve the equations $H_n = AF_{n-2} + BF_{n-1}$ and $H_{n-1} = AF_{n-3} + BF_{n-2}$ simultaneously for A and B. □

Now we can use the four lemmas above to find the S.G.F. sequence $\{H_n(A, B)\}$ with $N = H_R(A, B)$ such that $R = \rho(N)$, given any positive integer N. It is important to note that, if $B = 2A$, $\{H_n\}$ is a Fibonacci-like sequence and the maximal subscript R will increase by 1, since $H_n = AF_{n+1}$. Lemma 4.2 gives a test for a Fibonacci-like sequence, and a shortened solution since, if $|(-1)^n D| = K^2$, then $H_n = KF_{n+1}$.

Example 1: Let $N = 2001 = H_n$. Compute case (i): $[2001/\alpha] = 1236 = H_{n-1}$, and $(-1)^n D = 3069 > 0$, so n_1 is even; next, $F_9 < 2001/\sqrt{3069} \approx 36.1 \le F_{10}$, so $n_1 = 10$. Compute case (ii) using $[2001/\alpha] + 1 = 1237 = H_{n-1}$ and $(-1)^n D = -1405 < 0$, so n_2 is odd; with $F_9 < 2001/\sqrt{1405} \approx 53.38 \le F_{10}$, $n_2 = 9$. Take $R = \max(10, 9) = 10 = n$, and use $H_{n-1} = 1236$ from case (i) to compute $a = |1236F_9 - 2001F_8| = 3$, $b = |1236F_8 - 2001F_7| = 57$. Since $b > 2a$, take $N = H_{10}(3, 57)$.

Example 2: Let $N = 357 = H_n$. $[357/\alpha] = 220 = H_{n-1}$ and $(-1)^n D = 509 > 0$, so n_1 is even. Then $F_7 < 357/\sqrt{509} \approx 15.8 \le F_8$, so $n_1 = 8$. Compute case (ii) for $H_{n-1} = 221$, obtaining $(-1)^n D = -289 < 0$, so n_2 is odd; $F_7 < 357/\sqrt{289} = 21 \le F_8$, so $n_2 = 7$. We choose $n = n_1 = 8$ and use $H_{n-1} = 220$ to compute $a = |220F_7 - 357F_6| = 4$ and $b = |220F_6 - 357F_5| = 25$. Therefore, $R = 8$,

$A=4$, and $B=25$ yields $N=H_8(4,25)$. Note that $|(-1)^n D|289=17^2$ in case (ii) indicates a Fibonacci-like sequence, $n_2+1=8=R$, giving a double, and $H_{n-1}=221$ for $n_2=7$ yields $a=|221F_6-357F_5|=17=A$ and $b=|221F_5-357F_4|=34=B$, or $N=H_7(17,34)=17F_8$.

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REFERENCES

1. Brother Alfred Brousseau. *Introduction to Fibonacci Discovery*. Santa Clara, CA: The Fibonacci Association, 1965.
2. Brother U. Alfred (Brousseau). "On the Ordering of Fibonacci Sequences." *The Fibonacci Quarterly* **1.4** (1963):43-46.
3. Marjorie Bicknell-Johnson. "A Note on a Representation Conjecture by Hoggatt." In *Appli-cations of Fibonacci Numbers* **7**:39-42. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1998.
4. Marjorie Bicknell-Johnson & David A. Englund. "Greatest Integer Identities for Generalized Fibonacci Sequences $\{H_n\}$, Where $H_n=H_{n-1}+H_{n-2}$." *The Fibonacci Quarterly* **33.1** (1995): 50-58.
5. John H. E. Cohn. "Recurrent Sequences Including *N*." *The Fibonacci Quarterly* **29.1** (1991): 30-36.
6. L. E. Dickson. *History of the Theory of Numbers*. Vol. I. New York: Chelsea, 1971.
7. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969; rpt. Santa Clara, CA: The Fibonacci Association, 1979.
8. A. F. Horadam. "A Generalized Fibonacci Sequence." *Amer. Math. Monthly* **68.5** (1961): 455-59.
9. David A. Klarner. "Representations of *N* as a Sum of Distinct Elements from Special Sequen-ces." *The Fibonacci Quarterly* **4.4** (1966):289-306.
10. William J. LeVeque. *Elementary Theory of Numbers*, p. 32. London: Addison-Wesley, 1962.
11. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester, England: Ellis Horwood, 1989.

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