GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS AND MULTIPLICATIVE ARITHMETIC FUNCTIONS

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INTRODUCTION

In [7], two families of polynomials $\{F_{k,n}\}$ and $\{G_{k,n}\}$ in k indeterminates were defined:

$$
\begin{aligned}
&F_{k,0}(\mathbf{t}) = 0, && G_{k,0} = k,\\
&F_{k,1}(\mathbf{t}) = 1, && G_{k,1} = t_1,\\
&F_{k,n}(\mathbf{t}) = F_{k-1,n}(\mathbf{t}), 1 \le n \le k, && G_{k,n} = G_{k-1,n}, 1 \le n < k,\\
&F_{k,n}(\mathbf{t}) = \sum_{j=1}^{k} t_j F_{k,n-j}(t), n > k, && G_{k,n} = \sum_{j=1}^{k} t_j G_{k,n-j}, n \ge k,\\
&\text{where } \mathbf{t} = (t_1, \ldots, t_k).
\end{aligned}
$$

There it was pointed out that these two families generalize the Fibonacci and Lucas polynomials (see [3], e.g.). In the course of [7], they arose in a natural way in the context of a subgroup of the group of multiplicative arithmetic functions (see [8], e.g.); the group operation is the convolution product. The subgroup in question is sometimes called the *rational subgroup* of the group of *multiplicative functions* (RMF) (e.g., see [1]). It is the subgroup generated, under convolution by the *completely multiplicative functions* (CMF), those multiplicative functions γ which satisfy the identity $\gamma(m)\gamma(n) = \gamma(mn)\ \forall m, n \in \mathbb{N}$, where the product, this time, is the pointwise product. These CMF can also be described as those multiplicative functions which are completely determined by their values at primes. The RMF can be described as those multiplicative functions which are completely determined by their values on a finite number of prime powers for each prime p.

In [7], Corollary 1.3.2, it is shown that the rational group RMF is a(n uncountably generated) free abelian group.[1] The group minus the identity thus splits into two disjoint subsets, the free semigroup generated by the CMF's—call these the *positive* functions, and the set of their inverses—call these the *negative* functions. It is a consequence of the fact that elements are determined locally by their values on finitely many prime powers for each prime p, that there is associated with each pair consisting of a positive function χ and its inverse χ^{-1} a unique monic polynomial of degree k, $P_{\gamma,p}(\mathbf{t})$, $\mathbf{t} = (t_1, \ldots, t_k)$, with complex coefficients, and that k can be chosen to be the same for all primes p [7]; that is, the set of k's is bounded. Moreover, every such polynomial determines such a pair of rational MF's. An RMF determined in this way will be said to be of degree k. It is then clear that the positive functions form a graded semigroup.

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[1]A consequence of this result and a result of Carroll and Gioia [2] is that the rational group is embedded in a torsion-free divisible group in the group of multiplicative arithmetic functions.

The role of the (recursive) family of polynomials $\{F_{k,n}\}$ is that, when evaluated at the coefficients $(1, a_1, \ldots, a_k)$ of $P_{\gamma,p}(\mathbf{t})$, they give the values of γ at the n^{th} powers of the prime p. Thus, the set $\{F_{k,n}(\mathbf{t})\}$ yields every possible positive RMF of degree k under the evaluation map on k-tuples of complex numbers. A negative RMF (i.e., an inverse of a positive RMF) has a value 0 for all powers of p greater than k, and for powers of p less than or equal to k, the values are just the coefficients of $P_{\gamma,p}(\mathbf{t})$.

The polynomials $\{G_{k,n}(\mathbf{t})\}$ are somewhat more elusive, but are closely related to the $F_{k,n}$. When $k=2$ and $P_{\gamma,p}(\mathbf{t}) = x^2 - tx - 1$, then $F_{k,n}$ and $G_{k,n}$ are just the Fibonacci and Lucas polynomials, respectively. In general, $\partial G_{k,n} / \partial t_1 = nF_{k,n}$. From [7], we have the following generalization of the Binet formulas which, moreover, gives relations among the roots of $P_{\gamma,p}(x; t_1, \ldots, t_k)$, the values in the sequence $\gamma(p^n)$ and the polynomials $F_{k,n}(\mathbf{t})$. Thus, letting

$$\Delta_k = \Delta(\lambda_1, \ldots, \lambda_k) = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_k \\ \cdots & & \\ \lambda_1^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix}, \quad \Delta_{k,n} = \Delta_{k,n}(\lambda_1, \ldots, \lambda_k) = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_k \\ \cdots & & \\ \lambda_1^{k-2} & \cdots & \lambda_k^{k-2} \\ \lambda_1^{n+k-2} & \cdots & \lambda_k^{n+k-2} \end{vmatrix},$$

we have that

$$\gamma(p^n) = F_{k,n+1}(t) = \frac{\Delta_{k,n}}{\Delta_k},$$

where the $\lambda_j(\mathbf{t}) = \lambda_j$ are the k roots of the polynomial $P_{\gamma,p}(x; t_1, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k$. This is clearly a pleasant generalization of the Binet equations for $k=2$; but more is true.

$$\frac{\Delta_{k,n}}{\Delta_k} = \mathrm{CSP}(k,n) = \Sigma \lambda_1^{i_j} \cdots \lambda_k^{i_k}$$

where $\Sigma i_j = n$. These are just the *complete symmetric polynomials of degree n in the* λ_i ([6], pp. 21 ff.). The $G_{k,n}$ now become transparent: $G_{k,n}(\mathbf{t}(\lambda)) = \lambda_1^n + \cdots + \lambda_k^n = \mathrm{PSP}(k,n)$. These are just the *power symmetric polynomials of degree n in the* λ_j (see [6]).

In Section 1, Theorem 2, it is shown that each $F_{k,n+1}$ can be rewritten as a sum of products of the $G_{k,j}$ with rational coefficients; this rewriting process has an inverse which rewrites each of the $G_{k,n}$ as sums of products of the $F_{k,j}$ with integer coefficients (Theorem 3). There is also a map which sends $F_{k,n+1}$ and $G_{k,n}$ to symmetric polynomials in the roots of $P_\gamma(x; \mathbf{t})$, in the first case, a $\mathrm{CSP}(k,n)$, in the second, a $\mathrm{PSP}(k,n)$.

$$\begin{array}{ccc} F_{k,n+1} & \to & \mathrm{CSP}(k,n) \\ \downarrow & & \downarrow \\ G_{k,n} & \to & \mathrm{PSP}(k,n) \end{array}$$

All of these maps are invertible. This gives an effective process for rewriting the elements of the ring Λ^n of symmetric polynomials of degree N regarded as the Z-algebra generated by CPS's as elements in Λ^n regarded as the Q-algebra generated by the PSP's, and vice versa. In this way, the $F_{k,n}$ and the $G_{k,n}$ are identified with Schur polynomials in Λ^n.

There is also a number of relations among the two sets of polynomials $F_{k,n}$ and the $G_{k,n}$ which generalize the well-known relations among the Fibonacci and Lucas polynomials (e.g., see [4], or more conveniently [10], pp. 44-46). Those which have appeared in [7] will be listed for reference merely as Result without proof.

In 1995, Glasson [5] showed that the Fibonacci and Lucas polynomials satisfy second-order partial differential equations. We generalize this result in Theorem 4, Section 2. Here we show that the partial differential equation $D_{11} - \sum t_j D_{j2} = mD_2$, where $j = 1, \ldots, k$, is satisfied by $F_{k,n}$ if $m = 2$, and by $G_{k,n}$ if $m = 1$.

In Section 3, Theorem 5, we show that the RMF's are just those arithmetic functions which are locally recursive of finite degree.

1. IDENTITIES

Result 1 ([7], Theorem 3.4): $\dfrac{\partial G_{k,n}}{\partial t_1} = nF_{k,n},\ n \geq 0.$ □

Result 2 ([7], Corollary 3.4.1): $\displaystyle\sum_{j=1}^{k} \frac{\partial G_{k,n}}{\partial t_j} = n\sum_{j=0}^{k-1} F_{k,n-j}.$ □

Result 3 ([7], Corollary 3.4.2): $\displaystyle\sum_{j=1}^{k} \frac{\partial G_{k,n}}{\partial t_j} t_j = n\sum_{j=0}^{k-1} t_j F_{k,n-j} = nF_{k,n+1}.$ □

Result 4 ([7], Corollary 2.1.3 and Theorem 3.2): If the F-polynomials and G-polynomials are regarded as functions of the zeros of the defining monic polynomial $P_\gamma(x)$ of γ, then

(a) $F_{k,n}(\mathbf{t}(\boldsymbol{\lambda})) = \sum \lambda_1^{i_1} \cdots \lambda_k^{i_k}$, where $\sum i_j = n$.

(b) $G_{k,n}(\mathbf{t}(\boldsymbol{\lambda})) = \lambda_1^n + \cdots + \lambda_k^n$,

where $\mathbf{t} = (t_1, \ldots, t_k)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$. □

Theorem 1: $\displaystyle nF_{k,n+1} = \sum_{r=1}^{n} G_{k,r} F_{k,n-r+1}.$

First we prove

Lemma 1.1: $\displaystyle G_{k,0} = k,\ G_{k,n} = F_{k,n+1} + \sum_{j=1}^{k-1} j t_{j+1} F_{k,n-j},\ n \geq 1.$

Proof of Lemma 1.1: By definition, we can write

$$
\begin{aligned}
G_{k,n} = \sum_{j=1}^{k} t_j G_{k,n-j} &= \sum_{j=1}^{k} t_j \left[\left(\sum_{i=1}^{k-1} i t_{i+1} F_{k,n-j-i}\right) + F_{k,n-j+1}\right] \\
&= \sum_{j=1}^{k} t_j F_{k,n+1-j} + \sum_{i=1}^{k-1} \sum_{j=1}^{k} i t_j t_{i+1} F_{k,n-j-i},
\end{aligned}
$$

and again by definition,

$$
\begin{aligned}
&= F_{k,n+1} + \sum_{i=1}^{k-1} i t_{i+1} \left(\sum_{j=1}^{k} t_j F_{k,(n-i)-j}\right) \\
&= F_{k,n+1} + \sum_{i=1}^{k-1} i t_{i+1} F_{k,n-i}. \quad \square
\end{aligned}
$$

Proof of Theorem 1: Again, we proceed by induction noting that the result holds when $n=1$. We need to show that $(n+1)F_{k,n+2}=\sum_{r=1}^{n+1}G_{k,r}F_{k,n-r+2}$. With the understanding that $F_{k,m}=0$ when $m\le 0$, we have

$$\begin{aligned}
\sum_{r=1}^{n+1}G_{k,r}F_{k,n-r+2} &= \sum_{r=1}^{n}G_{k,r}\sum_{j=1}^{k}t_jF_{k,n-r-j+2}+G_{k,n+1}\\
&= \sum_{j=1}^{k}t_j\sum_{r=1}^{n}G_{k,r}F_{k,n-r-j+2}+G_{k,n+1}\\
&= \sum_{j=1}^{k}t_jF_{k,n+1-j}+\sum_{i=1}^{k-1}\sum_{j=1}^{k}it_jt_{i+1}F_{k,n-j-i}\\
&= F_{k,n+1}+\sum_{i=1}^{k-1}it_{i+1}\left(\sum_{j=1}^{k}t_jF_{k,(n-i)-j}\right)\\
&= \sum_{j=1}^{k}t_j(n-j+1)F_{k,n-j+2}+G_{k,n+1}\\
\text{so by Lemma 1.1}\quad &= \sum_{j=1}^{k}t_j(n-j+1)F_{k,n-j+2}+\sum_{j=1}^{k-1}jt_{j+1}F_{k,n-j+1}+F_{k,n+2}\\
&= F_{k,n+2}+t_k(n-k+1)F_{k,n-k+2}+\sum_{j=1}^{k-1}(t_j(n-j+1)F_{k,n-j+2}+jt_{j+1}F_{k,n-j+1})\\
&= F_{k,n+2}+n\sum_{j=1}^{k}t_jF_{k,n-j+2}=F_{k,n+2}+nF_{k,n+2}=(n+1)F_{k,n+2}.\ \square
\end{aligned}$$

The following two theorems give an effective rewriting process for writing products of PSP's in terms of CSP's and vice versa, that is, they will do so once it is explained how to write the PSP's and the CSP's in terms of the F- and G-polynomials. We shall state the theorems first.

Theorem 2:

$$F_{k,n+1}=\sum_{d_i\in d}\frac{1}{z_{d_i}}G_{k,i_1}\dots G_{k,i_{s_i}},\ z_{d_i}=\prod i_j^{\nu(i_j)}\nu(i_j)!$$

where $d=\{d_1,\dots,d_s\}$ is the set of partitions of n, $\nu(i_j)=$ number of times i_j occurs in $d_i=(i_1,\dots,i_{s(d)})$, $d_i\in d,\sum_{i=1}^{s}(1/z_{d_i})=1$.

Proof: Noting that the F-polynomials, when regarded as functions of the roots of the defining polynomial $P_\gamma(x)$ are just the complete symmetric polynomials; the G-polynomials are the power symmetric polynomials (Result 4), each of which is a basis for the space of symmetric polynomials. In particular, each polynomial $F_{k,r}$ can be written uniquely as a polynomial in the G-polynomials. So if the theorem is correct, it is just a statement of this representation. Now, $F_{k,n+1}$ regarded as a polynomial in the roots $\lambda_1,\dots,\lambda_k$, is complete symmetric of degree n; hence, each monomial summand is obtained as a partition of n; so in the language of Pólya's Counting Theorem [9], we let the figure inventory consist of $\lambda_1+\cdots+\lambda_k$ and then the cycle index is given by $(1/z_{d_i})G_{k,i_1}\dots G_{k,i_{s_i}}$, where $G_{k,r}=(\lambda_1^r+\cdots+\lambda_k^r)$. Since $i_1+\cdots+i_{s_i}=n$, $(1/z_{d_i})G_{k,i_1}\dots G_{k,i_{s_i}}$ is just a monomial of total degree n and so the sum is, indeed, $F_{k,n+1}$. Here, of course, $z_{d_i}=$ # conjugates of the element in S_n whose cycle structure is given by d_i. $\square$

Theorem 3:

$$G_{k,n} = n \sum_{d_i(n)} \frac{(-1)(l(d_i)-1)!}{\prod \nu_j(d_i)!} F_{k,i_2}, \ldots, F_{k,i_{s+1}},$$

using the notation of Theorem 2, and where $l(d_i) =$ length of d_i. □

The expressions in Theorems 2 and 3 are inverses of one another, which can be shown by direct computation, providing a proof of Theorem 3. Now, to get back and forth between the two sequences of polynomials, we identify the symbols t_j which appear in $G_{k,n}$ and $F_{k,n+1}$ with the elementary symmetric polynomials in the $\lambda_1, \ldots, \lambda_k$ as follows: $t_j = (-1)^{j+1}\sigma_{k,j}$, where $\sigma_{k,j} = \sigma_{k,j}(\lambda_1, \ldots, \lambda_k) =$ the j^{th} symmetric polynomial in the roots of the polynomial $P_{\gamma,p}(x; \mathbf{t})$. This identification is the basis of the proof of Lemma 4 in [7]. The substitution of the σ's for the t's yields the horizontal maps in the diagram in the introduction. The left-hand vertical arrows are just the maps implied by Theorems 1 and 2, Theorem 1 going downhill, Theorem 2 going uphill. For example, $\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2$ "is"

$$F_{2,3} = \tfrac{1}{2}G_{2,2} + \tfrac{1}{2}G_{2,1}^2 = \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2) + \tfrac{1}{2}(\lambda_1 + \lambda_2)^2 = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2;$$

$d_1 = [2]$, $d_2 = [1, 1]$.

2. PARTIAL DIFFERENTIAL EQUATIONS

Define a differential operator by $L_{2,m} = D_{11} - t_1 D_{12} - t_2 D_{22} - mD_2$, $m = 1, 2$, and, more generally, $L_{k,m} = D_{11} - \sum_{j=1}^{k} t_j D_{j2} - mD_2$. The following theorem states that the polynomials $G_{n,k}$ and $F_{n,k}$ are solutions of second-order partial differential equations, with the exceptions of the cases for $k = 1$.

Theorem 4: ***(a)*** $L_{k,1}G_{k,n} = 0, \quad k > 1,$

(b) $L_{k,2}F_{k,n} = 0, \quad k > 1.$

Proof: We proceed by induction.

Lemma: ***(a)*** $L_{2,1}G_{2,n} = 0,$

(b) $L_{2,2}F_{2,n} = 0.$

(These identities were proved in [5]; however, we shall give a proof here that is self-contained using the methods of this paper.) Assuming the result for $1 < r < n+1$, we can write $G_{2,n+1} = t_1 G_{2,n} + t_2 G_{2,n-1}$, and thus

$$\begin{aligned}
L_{2,1}(G_{2,n+1}) &= L_{2,1}(t_1 G_{2,n} + t_2 G_{2,n-1}) \\
&= t_1 L_{2,1}(G_{2,n}) + t_2 L_{2,1}(G_{2,n-1}) + 2D_1 G_{2,n} - t_1 D_2 G_{2,n} - t_1 D_1 G_{2,n-1} - 2t_2 D_2 G_{2,n-1} - G_{2,n-1} \\
&= 2nF_{2,n} - (2n-1)t_1 F_{2,n-1} - 2(n-1)t_2 F_{n-2} - F_{2,n} - t_2 F_{2,n-2} \\
&= (2n-1)F_{2,n} - (2n-1)F_{2,n} = 0,
\end{aligned}$$

equalities which follow from the induction hypothesis, definition of the F- and G- polynomials, and Result 1 and Lemma 1.1. □

The proof of part (b) follows from (a) and Result 1 as follows.

$$
\begin{aligned}
L_{2,2}(F_{2,n}) &= (D_{11} - t_1 D_{12} - t_2 D_{22} - 2D_2)F_{2,n} \\
&= (1/n)(D_{11} - t_1 D_{12} - t_2 D_{22} - 2D_2)D_1 G_{2,n} \\
&= 1/n\, D_1 L_{2,1} G_{2,n} + (D_{21} - D_{12})G_{2,n} = 0,
\end{aligned}
$$

using part (a). □

To complete the proof of the theorem, we assume that the result of the theorem holds for all $G_{s,n}$ for which $1 \le s \le k-1$, and note that $G_{k,n} = G_{k-1,n}$ for $1 \le n \le k$. Assume the result for $G_{k,s}$ for $1 \le s \le n$, and consider $L_{k,1} G_{k,n+1} = L G_{k,n+1}$, $n \ge k$. A straightforward, but rather tedious, computation, as in the proof of the lemma, using the inductive definition of $G_{k,n+1}$, which takes hold for this range of n's, and again using Result 1 and Lemma 1.1, and Theorems 2 and 3, we complete the induction. Part (b) now follows by a similar argument. □

3. CONCLUDING REMARKS

Theorem 5: Given the recursion $u_{j+1} = a_1 u_j + \cdots + a_k u_{j-k}$ with $u_0 = 0, u_1 = 1$, then

$$u_{j+1} = F_{k,j+1}(\mathbf{a}).$$

Proof: The theorem follows by induction and the definition. □

Notice that this result can be applied to any linear recursion formula, for if the coefficient of u_{j+1} is any nonzero (complex) number, we can divide through by it and apply the theorem.

We define a sequence to be *locally linearly recursive* of degree k if at each prime p the prime powers of the sequence are given by a linear recursive relation involving k independent unknowns, the same k for each prime p.

Corollary 5.1: A sequence is locally linearly recursive of degree k if and only if when regarded as an arithmetic function, it belongs to the positive semigroup of the group of rational multiplicative functions. □

We define a positive rational multiplicative sequence to be *uniform* if at each prime it is determined by the same polynomial, $P_{\gamma,p}(x) = P_{\gamma,p'}(x)$ for all primes p and p'. It is clear that the uniform sequences form a sub-semigroup of the semigroup of positive rational functions. It is also clear from the above corollary that

Corollary 5.2: A sequence is linearly recursive of degree k if and only if it is, as an arithmetic function, uniform. □

Here, *linear recursive of degree k* has the obvious meaning; the same relation holds for all primes.

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