SOME FURTHER PROPERTIES OF ANDRE-JEANNIN AND THEIR COMPANION POLYNOMIALS

M. N. S. Swamy
Concordia University, Montreal, Quebec, H3G 1M8, Canada
(Submitted May 1998-Final Revision January 1999)

1. INTRODUCTION

In a recent article [4], the author defined two sets of polynomials, $u_n(x)$ and $v_n(x)$, by the relations:

$$u_n(x) = (x+p)u_{n-1}(x) - qu_{n-2}(x), \quad n \geq 2, \tag{1.1a}$$

with

$$u_0(x) = 1, \; u_1(x) = x + p - \sqrt{q}, \tag{1.1b}$$

and

$$v_n(x) = (x+p)v_{n-1}(x) - qv_{n-2}(x), \quad n \geq 2, \tag{1.2a}$$

with

$$v_0(x) = 1, \; v_1(x) = x + p + \sqrt{q}, \tag{1.2b}$$

where $q > 0$, and showed that they are very closely related to two other sets of polynomials $U_n(x)$ and $V_n(x)$ defined by André-Jeannin (see [1] and [2]) by the relations

$$U_n(x) = (x+p)U_{n-1}(x) - qU_{n-1}(x), \quad n \geq 2, \tag{1.3a}$$

with

$$U_0(x) = 0, \; U_1(x) = 1, \tag{1.3b}$$

and

$$V_n(x) = (x+p)V_{n-1}(x) - qV_{n-2}(x), \quad n \geq 2, \tag{1.4a}$$

with

$$V_0(x) = 2, \; V_1(x) = x + p. \tag{1.4b}$$

In the same article, the author derived a few of the properties of the polynomials $u_n(x)$ and $v_n(x)$, as well as some interesting interrelationships. The purpose of this article is to derive further properties of these polynomials and their interrelationships. Since the modified Morgan-Voyce polynomials $\widetilde{B}_n(x)$, $\widetilde{b}_n(x)$, $\widetilde{c}_n(x)$, and $\widetilde{C}_n(x)$ defined in [4] result when $q = 1$, we thus derive a number of interesting properties of these modified Morgan-Voyce polynomials.

Since the polynomials $U_n(x)$ and $V_n(x)$ were defined and a number of their properties were studied for the first time by André-Jeannin, it is appropriate to refer to them as the André-Jeannin polynomials of the first and second kind. The polynomials $u_n(x)$ and $v_n(x)$, which are closely related to the André-Jeannin polynomials, and which exist as real distinct polynomials only when $q > 0$, will be referred to as the companion André-Jeannin polynomials of the first and second kind. We will now list a number of important properties of the polynomials $U_n(x)$, $u_n(x)$, $v_n(x)$, and $V_n(x)$ that are either known or easily derivable from the known properties, since these will be required in establishing the results of the remaining sections.

Simple Interrelations:

$$u_n(x) = U_{n+1}(x) - \sqrt{q}U_n(x), \qquad \text{from [4].} \tag{1.5}$$

$$v_n(x) = U_{n+1}(x) + \sqrt{q}U_n(x), \qquad \text{from [4].} \tag{1.6}$$

$$V_n(x) = U_{n+1}(x) - qU_{n-1}(x), \quad \text{from [2].} \tag{1.7}$$
$$V_n(x) = u_n(x) + \sqrt{q}u_{n-1}(x), \quad \text{from [4].} \tag{1.8}$$
$$V_n(x) = v_n(x) - \sqrt{q}v_{n-1}(x), \quad \text{from [4].} \tag{1.9}$$
$$(x+p-2\sqrt{q})U_n(x) = u_n(x) - \sqrt{q}u_{n-1}(x), \quad \text{by induction.} \tag{1.10}$$
$$(x+p-2\sqrt{q})v_n(x) = u_{n+1}(x) - qu_{n-1}(x), \quad \text{from (1.6) and (1.10).} \tag{1.11}$$
$$(x+p-2\sqrt{q})v_n(x) = V_{n+1}(x) - \sqrt{q}V_n(x), \quad \text{by induction.} \tag{1.12}$$

Simson Formulas:

$$U_{n+1}(x)U_{n-1}(x) - U_n^2(x) = -q^{n-1}, \quad \text{from [1],} \tag{1.13a}$$
$$u_{n+1}(x)u_{n-1}(x) - u_n^2(x) = q^{n-1/2}\Delta_u, \quad \text{from [4],} \tag{1.13b}$$
$$v_{n+1}(x)v_{n-1}(x) - v_n^2(x) = -q^{n-1/2}\Delta_v, \quad \text{from [4],} \tag{1.13c}$$
$$V_{n+1}(x)V_{n-1}(x) - V_n^2(x) = q^{n-1}\Delta_u\Delta_v, \quad \text{from [2],} \tag{1.13d}$$

where

$$\Delta_u = x+p-2\sqrt{q}, \tag{1.14a}$$
$$\Delta_v = x+p+2\sqrt{q}. \tag{1.14b}$$

Binet's Formulas:

$$U_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{from [1],} \tag{1.15a}$$
$$u_n(x) = \frac{\alpha^{n+1/2} + \beta^{n+1/2}}{\alpha^{1/2} + \beta^{1/2}}, \quad \text{from (1.5) and (1.15a),} \tag{1.15b}$$
$$v_n(x) = \frac{\alpha^{n+1/2} - \beta^{n+1/2}}{\alpha^{1/2} - \beta^{1/2}}, \quad \text{from (1.6) and (1.15a),} \tag{1.15c}$$
$$V_n(x) = \alpha^n + \beta^n, \quad \text{from [2],} \tag{1.15d}$$

where

$$\alpha + \beta = x + p, \ \alpha\beta = q, \tag{1.16a}$$
$$\alpha - \beta = \sqrt{\Delta}, \ \Delta = \Delta_u\Delta_v = (x+p)^2 - 4q. \tag{1.16b}$$

2. NEW INTERRELATIONSHIPS

In this section, we will give a number of interesting relations between the André-Jeannin polynomials $U_n(x)$, $V_n(x)$, and their companions $u_n(x)$, $v_n(x)$. In order to present the results in a compact form, we will denote by $A_n(x)$ any one of the polynomials $U_n(x)$, $u_n(x)$, $v_n(x)$, or $V_n(x)$. We will first establish the following Lemma concerning $A_n(x)$ that is extremely useful in establishing certain relations needed to derive the results given in Section 4.

Lemma 1:
$$A_n(x)U_{r-h+2}(x) = A_r(x)U_{n-h+2}(x) - q^{r-h+2}A_{h-2}(x)U_{n-r}(x). \tag{2.1}$$

Proof: We confine ourselves to establishing the result when $A_n(x) \equiv U_n(x)$. Using (1.15a), we have

$$U_n(x)U_{r-h+2}(x)-U_r(x)U_{n-h+2}(x)$$
$$=\frac{\alpha^n-\beta^n}{\alpha-\beta}\cdot\frac{\alpha^{r-h+2}-\beta^{r-h+2}}{\alpha-\beta}-\frac{\alpha^r-\beta^r}{\alpha-\beta}\cdot\frac{\alpha^{n-h+2}-\beta^{n-h+2}}{\alpha-\beta}$$
$$=\frac{-(\alpha\beta)^{r-h+2}}{(\alpha-\beta)^2}[\{\alpha^{n-r+h-2}+\beta^{n-r+h-2}\}-\{\alpha^{n-r}\beta^{h-2}+\beta^{n-r}\alpha^{h-2}\}]$$
$$=-(\alpha\beta)^{r-h+2}\cdot\frac{\alpha^{h-2}-\beta^{h-2}}{\alpha-\beta}\cdot\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}$$
$$=q^{r-h+2}U_{h-2}(x)U_{n-r}(x),\ \text{using (1.16a) and (1.15a).}$$

In a similar manner, Lemma 1 can be established when $A_n(x)\equiv u_n(x)$, $v_n(x)$, and $V_n(x)$ by using (1.15a) along with (1.15b), (1.15c), and (1.15d), respectively.

By letting $r=n-h+1$ and $h=n-r+1$ in (2.1), we get the following result:

$$A_n(x)U_{r-h+2}(x)=A_{n-h+1}(x)U_{r+1}(x)-q^{r-h+2}A_{n-r-1}(x)U_{h-1}(x). \tag{2.2}$$

Now, by equating the right-side expressions in (2.1) and (2.2) and rearranging, we get the determinantal relation

$$\begin{vmatrix} U_{r+1}(x) & U_{n-h+2}(x) \\ A_r(x) & A_{n-h+1}(x) \end{vmatrix} = q^{r-h+2}\begin{vmatrix} U_{h-1}(x) & U_{n-r}(x) \\ A_{h-2}(x) & A_{n-r-1}(x) \end{vmatrix}. \tag{2.3}$$

Now, letting $r=m$, $h=m-r+2$, and $n=m+n+1-r$ in (2.3), we get the interesting relation

$$\begin{vmatrix} U_{m+1}(x) & U_{n+1}(x) \\ A_m(x) & A_n(x) \end{vmatrix} = q^{r}\begin{vmatrix} U_{m+1-r}(x) & U_{n+1-r}(x) \\ A_{m-r}(x) & A_{n-r}(x) \end{vmatrix}. \tag{2.4}$$

Now letting $h=2$, $n=n+1$, and $r=n$ in (2.1), we have

$$U_{n+1}(x)A_n(x)-A_{n+1}(x)U_n(x)=q^nA_0(x). \tag{2.5}$$

Also, letting $m=r=n-1$ in (2.4), we get

$$U_{n+1}(x)A_{n-1}(x)-U_n(x)A_n(x)=q^{n-1}[(x+p)A_0(x)-A_1(x)]. \tag{2.6}$$

It is observed that when $A_n(x)\equiv U_n(x)$, (2.6) reduces to (1.13a). Now, by letting $n=m+n+1$, $h=n+2$, and $r=n+1$ in (2.1), we have the relation

$$A_{m+n+1}(x)=U_{m+1}(x)A_{n+1}(x)-qU_m(x)A_n(x). \tag{2.7}$$

Hence,

$$A_{2n+1}(x)=U_{n+1}(x)A_{n+1}(x)-qU_n(x)A_n(x) \tag{2.8}$$

and

$$A_{2n}(x)=U_n(x)A_{n+1}(x)-qU_{n-1}(x)A_n(x)=A_n(x)U_{n+1}(x)-qA_{n-1}(x)U_n(x). \tag{2.9}$$

We may derive a number of other interesting relations. However, we present only a few of these relations that will be useful in deriving the results of Section 4:

$$u_{n-1}(x)v_n(x)-u_n(x)v_{n-1}(x)=2q^{n-1/2}, \tag{2.10a}$$
$$u_{n-1}(x)V_n(x)-u_n(x)V_{n-1}(x)=-q^{n-1}\Delta_u, \tag{2.10b}$$

$$v_{n-1}(x)V_n(x) - v_n(x)V_{n-1}(x) = -q^{n-1}\Delta_v. \tag{2.10c}$$

$$\sum_{r=0}^{n} q^{n-r} A_{2r+1}(x) = A_{n+1}(x)U_{n+1}(x), \tag{2.11a}$$

$$\sum_{r=0}^{n} q^{n-r} A_{2r}(x) = A_n(x)U_{n+1}(x), \tag{2.11b}$$

$$\sum_{r=1}^{n} q^{n-r} A_{2r}(x) = U_n(x)A_{n+1}(x). \tag{2.11c}$$

In passing, it may be mentioned that, if we let $p = 2$, $q = 1$, and $x = 1$, we have

$$U_n(1) = F_{2n},\ u_n(1) = F_{2n+1},\ v_n(1) = L_{2n+1},\ V_n(1) = L_{2n}. \tag{2.12}$$

Using identities (2.1)-(2.11), we may derive a number of interesting identities for Fibonacci and Lucas numbers. One such identity is the following, which may be obtained by letting $A_n(1) = V_n(1)$ in (2.5):

$$F_{2(n+1)}L_{2n} - F_{2n}L_{2(n+1)} = 2. \tag{2.13}$$

3. DERIVATIVES AND DIFFERENTIAL EQUATIONS

We now derive formulas for the derivatives of $U_n(x)$, $u_n(x)$, $v_n(x)$, and $V_n(x)$ with respect to x.

Theorem 1:

$$U_n'(x) = \sum_{r=1}^{n-1} U_r(x)U_{n-r}(x). \tag{3.1}$$

Proof: We establish the theorem by induction. The result is easily verified to be true for $n = 1, 2,$ and 3. Now, assuming the theorem to be true for n and $n+1$, we have

$$\begin{aligned}
U_{n+2}'(x) &= (x+p)U_{n+1}'(x) - qU_n'(x) + U_{n+1}(x), \text{ using (1.3a)} \\
&= (x+p)\sum_{r=1}^{n} U_r(x)U_{n+1-r}(x) - q\sum_{r=1}^{n-1} U_r(x)U_{n-r}(x) + U_{n+1}(x) \\
&= \sum_{r=1}^{n-1} U_r(x)[(x+p)U_{n+1-r}(x) - qU_{n-r}(x)] + (x+p)U_n(x)U_1(x) + U_{n+1}(x) \\
&= \sum_{r=1}^{n-1} U_r(x)U_{n+2-r}(x) + U_n(x)U_2(x) + U_{n+1}(x) = \sum_{r=1}^{n+1} U_r(x)U_{n+2-r}(x).
\end{aligned}$$

Hence the theorem.

Corollary 1:

$$u_n'(x) = \sum_{r=1}^{n} U_r(x)u_{n-r}(x). \tag{3.2}$$

Proof:

$$\begin{aligned}
u_n'(x) &= U_{n+1}'(x) - \sqrt{q}U_n'(x), \text{ using (1.5),} \\
&= \sum_{r=1}^{n} U_r(x)U_{n+1-r}(x) - \sqrt{q}\sum_{r=1}^{n-1} U_r(x)U_{n-r}(x), \text{ from Theorem 1,}
\end{aligned}$$

$$= U_n(x)U_1(x) + \sum_{r=1}^{n-1} U_r(x)[U_{n+1-r}(x) - \sqrt{q}U_{n-r}(x)]$$

$$= U_n(x)u_0(x) + \sum_{r=1}^{n-1} U_r(x)u_{n-r}(x), \text{ from (1.5),}$$

$$= \sum_{r=1}^{n} U_r(x)u_{n-r}(x).$$

Corollary 2:

$$v_n'(x) = \sum_{r=1}^{n} U_r(x)v_{n-r}(x). \tag{3.3}$$

This corollary can be proved along the same lines as Corollary 1, using relation (1.6) and Theorem 1.

Corollary 3:

$$V_n'(x) = \sum_{r=1}^{n} U_r(x)V_{n-r}(x) - U_n(x). \tag{3.4}$$

This corollary can be established using (1.9) and Corollary 2.

It is also known that (see [2])

$$V_n'(x) = nU_n(x). \tag{3.5}$$

By induction, we may derive the following similar results for the derivatives of $u_n(x)$ and $v_n(x)$ in terms of $U_n(x)$.

Theorem 2:

$$(x+p+2\sqrt{q})u_n'(x) = nU_{n+1}(x) + \sqrt{q}(n+1)U_n(x). \tag{3.6}$$

Theorem 3:

$$(x+p-2\sqrt{q})v_n'(x) = nU_{n+1}(x) - \sqrt{q}(n+1)U_n(x). \tag{3.7}$$

In passing, it may be observed that, from (3.4) and (3.5), we have the following interesting relation:

$$\sum_{r=1}^{n} U_r(x)V_{n-r}(x) = (n+1)U_n(x). \tag{3.8}$$

André-Jeannin [3] has shown that $U_n(x)$ and $U_n(x)$ satisfy, respectively, the differential equations

$$U_n(x)\text{: } \Delta y'' + 3(x+p)y' - (n^2-1)y = 0 \tag{3.9}$$

and

$$V_n(x)\text{: } \Delta y'' + (x+p)y' - n^2 y = 0, \tag{3.10}$$

where Δ is given by (1.16b). We now establish similar differential equations satisfied by $u_n(x)$ and $v_n(x)$.

Theorem 4: The polynomial $u_n(x)$ satisfies the differential equation

$$\Delta y'' + 2(x+p-\sqrt{q})y' - n(n+1)y = 0.$$

Proof: Since $U_n(x)$ satisfies the differential equation given by (3.9), we have

$$\Delta U_{n+1}''(x) + 3(x+p)U_{n+1}'(x) - n(n+2)U_{n+1}(x) = 0 \tag{3.11}$$

and

$$\Delta U_n''(x)+3(x+p)U_n'(x)-(n^2-1)U_n(x)=0. \tag{3.12}$$

Multiplying (3.12) by $\sqrt{q}$, then subtracting it from (3.11) and making use of relation (1.5) in the resulting equation, we get

$$\Delta u_n''(x)+3(x+p)u_n'(x)-n(n+1)u_n(x)-[nU_{n+1}(x)+(n+1)\sqrt{q}U_n(x)]=0. \tag{3.13}$$

Use of Theorem 2 reduces (3.13) to

$$\Delta u_n''(x)+2(x+p-\sqrt{q})u_n'(x)-n(n+1)u_n(x)=0. \tag{3.14}$$

Hence the theorem.

Similarly, by using (1.6), (3.9), and Theorem 3, we can prove the following result regarding $v_n(x)$.

Theorem 5: The polynomial $v_n(x)$ satisfies the differential equation

$$\Delta y''+2(x+p+\sqrt{q})y'-n(n+1)y=0. \tag{3.15}$$

André-Jeannin [3] has further shown that $U_n^{(k)}(x)$ and $V_n^{(k)}(x)$, $k=0,1,2,\dots,$ where the superscript (k) stands for the k^{th} derivative with respect to x, satisfies the following differential equations:

$$U_n^{(k)}(x):\ \Delta y''+(2k+3)(x+p)y'+\{(k+1)^2-n^2\}y=0, \tag{3.16a}$$

$$V_n^{(k)}(x):\ \Delta y''+(2k+1)(x+p)y'+(k^2-n^2)y=0. \tag{3.16b}$$

Using a similar procedure, and using Theorems 4 and 5, we may also establish that $u_n^{(k)}(x)$ and $v_n^{(k)}(x)$ satisfy the following differential equations:

$$u_n^{(k)}(x):\ \Delta y''+2(k+1)(x+p-\sqrt{q})y'+\{k(k+1)-n(n+1)\}y=0, \tag{3.16c}$$

$$v_n^{(k)}(x):\ \Delta y''+2(k+1)(x+p+\sqrt{q})y'+\{k(k+1)-n(n+1)\}y=0. \tag{3.16d}$$

It may be pointed out that the above two differential equations are, respectively, the generalizations of the corresponding ones for the modified Morgan-Voyce polynomials $\tilde{b}_n(x)$ and $\tilde{c}_n(x)$ given in [4].

4. INTEGRAL PROPERTIES

From (3.5), we have the result

$$\int U_n(x)\,dx=\frac{V_n(x)}{n}+K. \tag{4.1}$$

Hence,

$$\int u_n(x)\,dx=\frac{V_{n+1}(x)}{n+1}-\sqrt{q}\frac{V_n(x)}{n}+K, \text{ from (1.5) and (4.1)}, \tag{4.2}$$

$$\int v_n(x)\,dx=\frac{V_{n+1}(x)}{n+1}+\sqrt{q}\frac{V_n(x)}{n}+K, \text{ from (1.6) and (4.1)}, \tag{4.3}$$

and

$$\int V_n(x)\,dx=\frac{V_{n+1}(x)}{n+1}-q\frac{V_{n-1}(x)}{n}+K, \text{ from (1.7) and (4.1)}. \tag{4.4}$$

Let us denote

$$a=-p-2\sqrt{q},\quad b=-p+2\sqrt{q}. \tag{4.5}$$

Then we can establish by induction that

$$V_n(a)=(-1)^n 2q^{n/2},\quad V_n(b)=2q^{n/2}. \tag{4.6}$$

Using (4.6) in (4.1), (4.2), (3.3), and (4.4), we have the following results:

$$\int_a^b U_{2n}(x)\,dx=0, \tag{4.7a}$$

$$\int_a^b U_{2n+1}(x)\,dx=\frac{4}{2n+1}q^{n+1/2}, \tag{4.7b}$$

$$\int_a^b u_{2n}(x)\,dx=-\int_a^b u_{2n+1}(x)\,dx=\frac{4}{2n+1}q^{n+1}, \tag{4.8}$$

$$\int_a^b v_{2n}(x)\,dx=\int_a^b v_{2n+1}(x)\,dx=\frac{4}{2n+1}q^{n+1}, \tag{4.9}$$

$$\int_a^b V_{2n}(x)\,dx=-\frac{8}{4n^2-1}q^{n+1/2}, \tag{4.10a}$$

$$\int_a^b V_{2n+1}(x)\,dx=0. \tag{4.10b}$$

Letting $A_n(x)\equiv U_n(x)$ in (2.11b) and using (4.7a), we see that

$$\int_a^b U_{n+1}(x)U_n(x)\,dx=0. \tag{4.11a}$$

Also, by letting $A_n(x)\equiv U_n(x)$ in (2.11a) and using (4.7b), we have

$$\int_a^b U_{n+1}^2(x)\,dx=\sum_{r=0}^{n} q^{n-r}\frac{4q^{r+1/2}}{2r+1}=\sum_{r=0}^{n}\frac{4q^{n+1/2}}{2r+1}.$$

Hence,

$$\int_a^b U_n^2(x)\,dx=4q^{n-1/2}\sum_{r=0}^{n}\frac{1}{2r+1}. \tag{4.11b}$$

Now, integrating (1.13a) and using (4.11b), we have

$$\int_a^b U_{n+1}(x)U_{n-1}(x)\,dx=4q^{n-1/2}\sum_{r=0}^{n-1}\frac{1}{2r+1}. \tag{4.11c}$$

Similarly, by successively letting $A_n(x)\equiv u_n(x)$, $v_n(x)$, and $V_n(x)$ in (2.11c), (2.11a), and (2.11b) and using (4.8), (4.9), (4.10a), and (4.10b), we can derive the following relations:

$$\int_a^b u_{n+1}(x)U_n(x)\,dx=\int_a^b v_{n+1}(x)U_n(x)\,dx=4q^{n+1/2}\sum_{r=1}^{n}\frac{1}{2r+1}, \tag{4.12a}$$

$$\int_a^b V_{n+1}(x)U_n(x)\,dx=-\frac{8n}{2n+1}q^{n+1/2}, \tag{4.12b}$$

$$\int_a^b u_n(x)U_n(x)\,dx=-\int_a^b v_n(x)U_n(x)\,dx=-4q^n\sum_{r=0}^{n-1}\frac{1}{2r+1}, \tag{4.13a}$$

$$\int_a^b V_n(x)U_n(x)\,dx=0, \tag{4.13b}$$

$$\int_a^b u_n(x)U_{n+1}(x)\,dx = \int_a^b v_n(x)U_{n+1}(x)\,dx = 4q^{n+1/2}\sum_{r=0}^{n}\frac{1}{2r+1}, \tag{4.14a}$$

$$\int_a^b V_n(x)U_{n+1}(x)\,dx = \frac{8(n+1)}{2n+1}q^{n+1/2}. \tag{4.14b}$$

Corresponding to the relations (4.11a), (4.11b), and (4.11c) for $U_n(x)$, we may derive relations for the polynomials $u_n(x)$, $v_n(x)$, and $V_n(x)$. Substituting (1.5) and (1.6), respectively, for $u_n(x)$ and $v_n(x)$ in the expressions $u_{n+1}(x)u_n(x)$ and $v_{n+1}(x)v_n(x)$, and utilizing the relations (4.12a) and (4.13a), we have

$$\int_a^b u_{n+1}(x)u_n(x)\,dx = -\int_a^b v_{n+1}(x)v_n(x)\,dx = -4q^{n-1}\left[1+2\sum_{r=1}^{n}\frac{1}{2r+1}\right]. \tag{4.15a}$$

Substituting (1.5) and (1.6) in the expressions $u_n^2(x)$ and $v_n^2(x)$ and using (4.11a) and (4.11b), we get

$$\int_a^b u_n^2(x)\,dx = \int_a^b v_n^2(x)\,dx = 4q^{n+1/2}\left[\frac{1}{2n+1}+2\sum_{r=0}^{n-1}\frac{1}{2r+1}\right]. \tag{4.15b}$$

Now, integrating both sides of (1.13b) and (1.13c) and using (4.15b), we derive

$$\int_a^b u_{n+1}(x)u_{n-1}(x)\,dx = \int_a^b v_{n+1}(x)v_{n-1}(x)\,dx = 4q^{n+1/2}\left[\frac{1}{2n+1}+2\sum_{r=1}^{n-1}\frac{1}{2r+1}\right]. \tag{4.15c}$$

The corresponding expressions involving $V_n(x)$ may be derived using (1.8), (1.9), (1.13d), (4.15a), (4.15b), and (4.15c). These are:

$$\int_a^b V_{n+1}(x)V_n(x)\,dx = 0, \tag{4.16a}$$

$$\int_a^b V_n^2(x)\,dx = \frac{16(2n^2-1)}{(4n^2-1)}q^{n+1/2}, \tag{4.16b}$$

$$\int_a^b V_{n+1}(x)V_{n-1}(x)\,dx = -\frac{16(2n^2+1)}{3(4n^2-1)}q^{n+1/2}. \tag{4.16c}$$

In a similar manner, we can derive relations regarding integrals involving $u_n(x)$ and $v_n(x)$, $u_n(x)$ and $V_n(x)$, and $v_n(x)$ and $V_n(x)$. These correspond to relations (4.12a), (4.12b), and (4.12c) which, respectively, involve $u_n(x)$ and $U_n(x)$, $v_n(x)$ and $U_n(x)$, and $V_n(x)$ and $U_n(x)$. These are:

$$\int_a^b u_n(x)v_{n+1}(x)\,dx = -\int_a^b v_n(x)u_{n+1}(x)\,dx = 4q^{n+1}, \tag{4.17a}$$

$$\int_a^b u_n(x)v_n(x)\,dx = \frac{4}{2n+1}q^{n+1/2}, \tag{4.17b}$$

$$\int_a^b u_n(x)V_{n+1}(x)\,dx = -\int_a^b v_n(x)V_{n+1}(x)\,dx = \frac{8n}{2n+1}q^{n+1}, \tag{4.18a}$$

$$\int_a^b u_n(x)V_n(x)\,dx = \int_a^b v_n(x)V_n(x)\,dx = \frac{8(n+1)}{2n+1}q^{n+1/2}, \tag{4.18b}$$

$$\int_a^b u_{n+1}(x)V_n(x)\,dx = -\int_a^b v_{n+1}(x)V_n(x)\,dx = \frac{8(n+1)}{2n+1}q^{n+1}. \tag{4.18c}$$

ACKNOWLEDGMENT

I would like to thank the anonymous referee for many helpful suggestions and comments that improved the presentation of this paper.

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AMS Classification Numbers: 11B39, 26A06

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