ON p-ADIC COMPLEMENTARY THEOREMS BETWEEN PASCAL'S TRIANGLE AND THE MODIFIED PASCAL TRIANGLE*

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(Submitted July 1996-Final Revision January 2000)

1. INTRODUCTION

For any entry X inside Pascal's triangle, there are six entries next to and surrounding X which form a hexagon $A_1A_2A_3A_4A_5A_6$ (taken counterclockwise in this order).

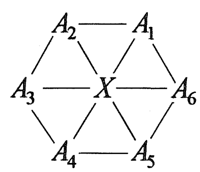

FIGURE 1

H. W. Gould [3] conjectured that an equal GCD property of the values of the binomial coefficients, namely

$$\mathrm{GCD}(A_1, A_3, A_5) = \mathrm{GCD}(A_2, A_4, A_6),$$

is true for all choices of the central entry X on Pascal's triangle. Gould called this conjecture a Star of David equality.

This equality was proved p-adically first by A. P. Hillman and V. E. Hoggatt, Jr. [4], and then by many others. A simple non-p-acid proof for the Star of David equality is given by S. Hitotumatu and D. Sato [5].

It is clear that an analogous equal LCM property for the Star of David configuration, namely

$$\mathrm{LCM}(A_1, A_3, A_5) = \mathrm{LCM}(A_2, A_4, A_6),$$

does not hold on Pascal's triangle.

In order to obtain an analogous LCM equality for two triplets $\{A_1, A_3, A_5\}$ and $\{A_2, A_4, A_6\}$, S. Ando [1] proposed a modified number array that has modified binomial coefficients $X' = (n+1)!/k!(n-k)!$ as its entries instead of binomial coefficients $X = n!/k!(n-k)!$ and called it the modified Pascal triangle. The beginning parts of Pascal's triangle and the modified Pascal triangle corresponding to $0 \le n \le 6$ are shown in Figure 2.

This modified Pascal triangle consists of the reciprocals of the entries on the harmonic triangle of Leibniz which has been studied by G. W. Leibniz as a method of summing up an infinite telescopic sequence.

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* The content of this paper was presented to The Fibonacci Association under the title "On p-adic Duality between Pascal's Triangle and the Harmonic Triangle I" at the Seventh International Research Conference on Fibonacci Numbers and Their Applications held at the Technische Universität in Graz, Austria, on July 15-19, 1996.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

(a) Pascal's Triangle $\Delta(B)$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 2 & & 2 & & & & & \\
 & & & & 3 & & 6 & & 3 & & & & \\
 & & & 4 & & 12 & & 12 & & 4 & & & \\
 & & 5 & & 20 & & 30 & & 20 & & 5 & & \\
 & 6 & & 30 & & 60 & & 60 & & 30 & & 6 & \\
7 & & 42 & & 105 & & 140 & & 105 & & 42 & & 7
\end{array}$$

(b) Modified Pascal Triangle $\Delta(M)$

FIGURE 2

On the modified Pascal triangle, the equal LCM property of the new Star of David configuration, namely

$$\mathrm{LCM}(A'_1, A'_3, A'_5) = \mathrm{LCM}(A'_2, A'_4, A'_6),$$

always holds, no matter where we take the center X'.

While the equal LCM property holds on this modified Pascal triangle, it is easy to see that the equal GCD property of two triplets $\{A'_1, A'_3, A'_5\}$ and $\{A'_2, A'_4, A'_6\}$, namely

$$\mathrm{GCD}(A'_1, A'_3, A'_5) = \mathrm{GCD}(A'_2, A'_4, A'_6),$$

no longer holds there.

Moreover, we studied in [2] a necessary and sufficient condition that rays of a star configuration on Pascal's triangle or on the modified Pascal triangle cover its center with respect to GCD and LCM. We do not want to repeat the results here, but the conditions for GCD and LCM on Pascal's triangle correspond to those for LCM and GCD on the modified Pascal triangle, respectively, although on the modified Pascal triangle we have to take the reflection of configurations on Pascal's triangle with respect to the horizontal line (see item (i) of Section 2 and the Corollary in Section 4).

The purpose of this paper is to clarify the reason why such a phenomenon occurs between these triangular arrays of numbers by showing a p-adic complementary relation of binomial coefficients and modified binomial coefficients.

2. DEFINITIONS, NOTATIONS, AND CLARIFICATIONS

(a) We denote the value of binomial coefficients as

$$X = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for $0 \le k \le n$. The triangular array of binomial coefficients is Pascal's triangle, which we denote by $\Delta(B)$.

(b) We call

$$\left\{ {n \atop k} \right\} = \frac{(n+1)!}{k!(n-k)!} = (n+1)\binom{n}{k}$$

the modified binomial coefficients, and we refer to a similar triangular array of these coefficients as the modified Pascal triangle, which we denote by $\Delta(M)$.

(c) Given a prime number p and an integer y, the additive p-adic valuation of y, denoted by $\beta = v_p(y)$, is the largest β such that p^β divides y.

Let the symbols $\binom{n}{k}$ and $\left\{ n \atop k \right\}$ represent both their numerical values and their positions on $\Delta(B)$ and $\Delta(M)$, respectively. The inequality $0 \le k \le n$ is assumed throughout the arguments.

(d) The set of both triangular arrays of nonnegative integers defined in (a) and (b) is denoted by $\Delta(T)$. Thus, $\Delta(T) = \{\Delta(B), \Delta(M)\}$, and $Y \in \Delta(T)$ means that Y is one of these two triangular arrays.

(e) A finite subset C of $Y \in \Delta(T)$ is called a configuration on Y.

(f) We introduce an equivalence relation to the set of all the configurations on Y such that two configurations on Y are equivalent to each other if and only if one is obtained by a parallel translation of the other. Then an equivalence class of the set of all configurations on Y by this equivalence relation is called a translatable configuration on Y. Unless otherwise stated, we simply call it a configuration C even if it is actually referring to the translatable configuration to which the configuration C belongs. There will not be a danger of misinterpretation since we are discussing only the GCD and LCM properties that hold on C independent of the location of C on Y.

(g) Let S_1 and S_2 be two nonempty finite subsets of $Y \in \Delta(T)$ and put $C = S_1 \cup S_2$. Then C is a configuration on Y. We do not claim $S_1 \cap S_2 = \phi$.

If the equality

$$\mathrm{GCD}(S_1) = \mathrm{GCD}(S_2) \tag{1}$$

holds independent of the location of C on Y, we call (1) a GCD equality on Y. In the case $S_1 = \{A_1, A_3, A_5\}$ and $S_2 = \{A_2, A_4, A_6\}$, (1) turns out to be the original Star of David equality. In the same manner, if

$$\mathrm{LCM}(S_1) = \mathrm{LCM}(S_2) \tag{2}$$

holds instead of (1), we call (2) a LCM equality in Y.

(h) The central symmetric axis of $Y \in \Delta(T)$ is the straight line of entries with $n = 2k$, where $k = 0, 1, 2, \ldots$, on Y. Any line of entries that is parallel to it on y is called a vertical line of Y. Y is supposed to be placed in the traditional way so that these lines are vertical. A set of entries with $n =$ constant on Y is called a horizontal line of Y. It is perpendicular to a vertical line of Y.

(i) We consider a group $K = \{I, V, H, R\}$ of transformations that operate on the configuration C on Y. I is the identity transformation by which each entry in C stays unchanged. V is the vertical reflection of C by which each entry in C moves to its symmetrical point with respect to a vertical line. H is the horizontal reflection of C by which each entry in C moves to its symmetrical point with respect to a horizontal line. R is a 180° rotation about a point X by which each entry in C moves to its symmetrical point with respect to X. X is not always a point in C, but sometimes is a midpoint of two entries on Y.

Notice that each transformation in K operates on C, not on Y, and we do not have to locate the reflection axis or the center of symmetry since we assume that configuration C on which each element of K operates is translatable.

(j) Group K is Klein's four group with unit I, and its elements satisfy the relations

$$V^2 = H^2 = R^2 = I, \quad VH = HV = R, \quad VR = RV = H, \quad HR = RH = V.$$

The images of a configuration C under the transformations V, H, and R are also called a vertical (or right-left) reflection of C, a horizontal (or upside-down) reflection of C and a 180° rotation of C, and are denoted by $V(C)$, $H(C)$, and $R(C)$, respectively.

3. p-ADIC COMPLEMENTARY THEOREM BETWEEN BINOMIAL AND MODIFIED BINOMIAL COEFFICIENTS

First, we will write a preparatory lemma concerning binomial coefficients.

Lemma 1: Let p be a given prime number and r be a nonnegative integer. Then we have

$$v_p\left(\binom{p^r-1}{k}\right) = 0$$

for $0 \le k \le p^r - 1$, and

$$v_p\left(\binom{2p^r-1}{k}\right) = 0$$

for $0 \le k \le 2p^r - 1$.

Proof: Both equalities are special cases of Theorem 8 in C. T. Long [6]. Notice that, for $p = 2$, the second equality is reduced to the first one.

Now, we will show our main result.

Theorem 1: Let p be a prime number and r be a nonnegative integer. Then, for any integers m, n, h, and k satisfying

$$m+n = 2p^r - 2, \quad h+k = p^r - 1, \quad 0 \le k \le n \text{ and } 0 \le h \le m, \tag{3}$$

we have

$$v_p\left(\binom{m}{h}\right) + v_p\left(\left\{ {n \atop k} \right\}\right) = r. \tag{4}$$

Proof: Using given conditions (3) and Lemma 1, we can easily show that

$$\begin{aligned}
v_p\left(\binom{m}{h}\right) + v_p\left(\left\{ {n \atop k} \right\}\right) &= v_p\left(\binom{m}{h}\left\{ {n \atop k} \right\}\right) = v_p\left(\frac{m!}{h!(m-h)!} \times \frac{(n+1)!}{k!(n-k)!}\right) \\
&= v_p\left(\frac{(h+k+1)\times(m+n+1)!}{(h+k+1)!(m+n-h-k)!} \times \frac{(h+k)!}{h!k!} \times \frac{(m+n-h-k)!}{(m-h)!(n-k)!} \div \frac{(m+n+1)!}{m!(n+1)!}\right) \\
&= v_p(h+k+1) + v_p\left(\binom{m+n+1}{h+k+1}\right) + v_p\left(\binom{h+p}{h}\right) + v_p\left(\binom{m+n-h-k}{m-h}\right) - v_p\left(\binom{m+n+1}{m}\right) \\
&= v_p(p^r) + v_p\left(\binom{2p^r-1}{h+k+1}\right) + v_p\left(\binom{p^r-1}{h}\right) + v_p\left(\binom{p^r-1}{m-h}\right) - v_p\left(\binom{2p^r-1}{m}\right) \\
&= r + 0 + 0 + 0 - 0 = r.
\end{aligned}$$

4. GCD-LCM DUALITY BETWEEN PASCAL'S TRIANGLE AND THE MODIFIED PASCAL TRIANGLE

As an application of the p-adic complementary theorem between the binomial coefficients and the modified binomial coefficients that was established in the previous section, we now prove a duality between Pascal's triangle $\Delta(B)$ and the modified Pascal triangle $\Delta(M)$ concerning the GCD and the LCM.

Let S_1 and S_2 be two nonempty finite subsets of $\Delta(B)$ and put $C = S_1 \cup S_2$. Then C is a configuration in $\Delta(B)$. First, we assume that

$$\mathrm{GCD}(S_1) = \mathrm{GCD}(S_2) \tag{5}$$

holds independent of the location of C in $\Delta(B)$.

Define $\min\{v_p(S)\}$ to be $\min\{v_p(A) \mid A \in S\}$. Then the GCD equality (5) is equivalent to

$$\min\{v_p(S_1)\} = \min\{v_p(S_2)\} \text{ for all primes } p. \tag{6}$$

Let R be a $180°$ rotation about a point X defined in item (i) in Section 2. Since we are discussing the translatable properties of the configurations, we can take any point X such that, for any entry $A \in \Delta(B)$, $R(A)$ is also an entry of $\Delta(B)$ as long as $R(A) \in \Delta(B)$.

We overlap two triangles $\Delta(B)$ and $\Delta(M)$ in such a way that $\binom{n}{k}$ and $\left\{ {n \atop k} \right\}$ fall on the same point. Then a configuration C in $\Delta(B)$ is also considered to be one in $\Delta(M)$, which is geometrically the same as C in $\Delta(B)$ although they are different as sets of integers. If $R(C) \subset \Delta(M)$, we put $C' = R(C)$. Then $C' = S_1' \cup S_2'$, where $S_1' = R(S_1)$ and $S_2' = R(S_2)$ are subsets of C'.

Let p be an arbitrary, but fixed prime. If we take the midpoint of

$$\binom{0}{0} \text{ and } \binom{2p^r - 2}{p^r - 1},$$

where r is a sufficiently large positive integer, as the center X of rotation R, then the configuration C corresponding to C' by R is contained in $\Delta(B)$. Any entry $A' \in C'$ and the corresponding entry $A \in C$ satisfies condition (3) of Theorem 1 if we let

$$A = \binom{m}{h} \text{ and } A' = \left\{ {n \atop k} \right\}.$$

Therefore, we have $v_p(A) + v_p(A') = r$ by Theorem 1, so that

$$\min\{v_p(S)\} + \max\{v_p(S')\} = r \tag{7}$$

for any $S' \in C'$ and corresponding $S \in C$.

Since we assume the GCD equality (5) on $\Delta(B)$, equality (6) holds so that, using (7), we have

$$\max\{v_p(S_1')\} = \max\{v_p(S_2')\} \text{ for all primes } p, \tag{8}$$

which is equivalent to

$$\mathrm{LCM}(S_1') = \mathrm{LCM}(S_2'). \tag{9}$$

Thus, (9) holds independent of the location of C' on $\Delta(M)$.

In a similar manner, we can prove that if (9) holds independent of the location of C' on $\Delta(M)$, then (5) holds independent of the location of C' on $\Delta(B)$. If we exchange min and max in

(6), (7), and (8), then GCD and LCM in (5) and (9) must be exchanged. Summarizing these arguments, we have the following results.

Theorem 2: Let $C = S_1 \cup S_2$ be a configuration on $\Delta(B)$ and $C' = S_1' \cup S_2'$ be the configuration on $\Delta(M)$ corresponding to C by a $180°$ rotation R about a point X. Then GCD equality (5) holds independent of the location of C on $\Delta(B)$ if and only if LCM equality (9) holds independent of the location of $C' = R(C)$ on $\Delta(M)$. Similarly, the LCM equality holds for C on $\Delta(B)$ if and only if the corresponding GCD equality for C' holds on $\Delta(M)$.

Corollary: Let $C = S_1 \cup S_2$ be as above and C'' be the configuration on $\Delta(M)$ corresponding to C by horizontal reflection H with respect to a horizontal line. If we put $H(S_1) = S_1''$, $H(S_2) = S_2''$, then $C'' = S_1'' \cup S_2''$. GCD equality (5) holds independent of the location of C on $\Delta(B)$ if and only if LCM equality $\mathrm{LCM}(S_1'') = \mathrm{LCM}(S_2'')$ holds independent of the location of $C'' = H(C)$ on $\Delta(M)$. Similarly, the LCM equality holds for C on $\Delta(B)$ if and only if the corresponding GCD equality for C'' holds on $\Delta(M)$.

Proof: A horizontal reflection H can be expressed as $H = RV$ by a vertical reflection V and a $180°$ rotation R. If we remember that the equality

$$\binom{n}{n-k} = \binom{n}{k}$$

holds for nonnegative integers n, k with $k \le n$, it is clear that a GCD equality or an LCM equality holds for C if and only if it holds for $V(C)$. Combining this fact with Theorem 2, we have the stated conclusion.

5. p-ADIC COMPLEMENTARY THEOREMS BETWEEN GENERALIZED BINOMIAL COEFFICIENTS AND GENERALIZED MODIFIED BINOMIAL COEFFICIENTS WHICH ARE DEFINED BY A STRONG DIVISIBILITY SEQUENCE

A sequence of integers $A = \{a_n\} = \{a_1, a_2, a_3, \dots\}$ is called a strong divisibility sequence if $(a_k, a_h) = a_{(k,h)}$ for every k, $h = 1, 2, 3, \dots$, where (a_k, a_h) and (k, h) are the greatest common divisors of the two numbers.

The sequence of natural numbers $\mathrm{N} = \{1, 2, 3, \dots\}$ and the sequence of Fibonacci numbers $F = \{F_1, F_2, F_3, \dots\}$ are two examples of strong divisibility sequences.

For any strong divisibility sequence $A = \{a_n\}$, if we generalize the binomial coefficients $\binom{n}{k}$ and modified binomial coefficients $\left\{{n \atop k}\right\}$ by replacing n and k in 2(a) and 2(b) by a_n and a_k throughout, then we have A-binomial coefficients

$$\binom{n}{k}_A = \frac{a_1 a_2 \dots a_n}{(a_1 a_2 \dots a_k)(a_1 a_2 \dots a_{n-k})}$$

and A-modified binomial coefficients

$$\left\{{n \atop k}\right\}_A = \frac{a_1 a_2 \dots a_n a_{n+1}}{(a_1 a_2 \dots a_k)(a_1 a_2 \dots a_{n-k})} = a_{n+1}\binom{n}{k}_A.$$

It is not difficult to obtain generalizations of the p-adic complementary theorem and the GCD-LCM duality theorem between analogous two-dimensional number arrays of A-binomial coefficients and A-modified binomial coefficients, both of which are defined by the same strong divisibility sequence $A = \{a_n\}$.

Those generalizations, other extensions, and their applications will be reported in subsequent papers in due course.

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AMS Classification Numbers: 11B65, 11A05, 05A10

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