ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-564 ***Proposed by Stanley Rabinowitz, Westford, MA***

Let k be a positive integer and let $a_0 = 1$. Find integers $a_1, a_2, \dots, a_k$ and $b_0, b_1, b_2, \dots, b_k$ such that

$$\sum_{i=0}^{k} a_i L_{n+i}^{2k} = \sum_{i=0}^{k} b_i F_{n+i}^{2k}$$

is true for all integers n. Prove that your answer is unique.

For example, when $k = 4$, we have the identity

$$L_n^8 + 21L_{n+1}^8 + 56L_{n+2}^8 + 21L_{n+3}^8 + L_{n+4}^8 = 625(F_n^8 + 21F_{n+1}^8 + 56F_{n+2}^8 + 21F_{n+3}^8 + F_{n+4}^8).$$

H-565 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Let p be a prime with $p \equiv -1 \pmod{2m}$, where $m \geq 3$ is an odd integer. Prove that all residues are m^{th} powers (mod p).

H-566 ***Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada***

Let $\phi_n := \pi / 2n$, where n is a positive integer, and set $L_n = a^n + b^n$, $F_n = (a^n - b^n)/(a-b)$, where $a = \frac{1}{2}(u + \sqrt{u^2 - 4})$, $b = \frac{1}{2}(u - \sqrt{u^2 - 4})$, $u \neq \pm 2$, and show that, for $n \geq 2$,

$$\begin{aligned} S_n(u) &:= \sum_{k=1}^{n-1} \frac{1}{1 + \left(\frac{u+2}{u-2}\right) tg^2(k\phi_n)} \\ &= -\frac{1}{2} + \frac{n}{2(u+2)^2 F_n}[L_{n+1} + 3L_n + 3L_{n-1} + L_{n-2}]. \end{aligned}$$

SOLUTIONS

A Very Odd Problem

H-550 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 2, May 1999)

Suppose n is an odd integer, p an odd prime $\neq 5$. Prove that $L_n \equiv 1 \pmod{p}$ if and only if (i) $\alpha^n \equiv \alpha$, $\beta^n \equiv \beta \pmod{p}$, or (ii) $\alpha^n \equiv \beta$, $\beta^n \equiv \alpha \pmod{p}$.

Solution by H.-J. Seiffert, Berlin, Germany

Recall that the integers of $Q(\sqrt{5})$ have the form $(a+\sqrt{5}b)/2$, where $a, b \in \mathbf{Z}$ such that $a \equiv b \pmod 2$.

Since n is odd, we have $L_n^2 - 5F_n^2 = -4$. Clearly, $L_n \equiv F_n \pmod 2$.

Suppose that $L_n \equiv 1 \pmod p$. Then $F_n^2 \equiv 1 \pmod p$, because p is a prime $\neq 5$. Hence, since p is an odd prime, either $F_n \equiv 1 \pmod p$ or $F_n \equiv -1 \pmod p$. If $F_n \equiv 1 \pmod p$, then

$$\alpha^n = \frac{L_n + \sqrt{5}F_n}{2} \equiv \frac{1+\sqrt{5}}{2} = \alpha \pmod p \text{ and } \beta^n = \frac{L_n - \sqrt{5}F_n}{2} \equiv \frac{1-\sqrt{5}}{2} = \beta \pmod p.$$

Similarly, if $F_n \equiv -1 \pmod p$, then

$$\alpha^n = \frac{L_n + \sqrt{5}F_n}{2} \equiv \frac{1-\sqrt{5}}{2} = \beta \pmod p \text{ and } \beta^n = \frac{L_n - \sqrt{5}F_n}{2} \equiv \frac{1+\sqrt{5}}{2} = \alpha \pmod p.$$

Conversely, suppose that either (i) or (ii) holds, then in each case $L_n = \alpha^n + \beta^n \equiv \alpha + \beta = 1 \pmod p$.

Also solved by L. A. G. Dresel and the proposer.

Some Restriction!

H-551 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 37, no. 2, May 1999)***

Let k be a nonnegative integer and define the following restricted double-sum,

$$S_k := \sum_{\substack{r=0 \\ br+as<ab}}^{a-1} \sum_{s=0}^{b-1} (br+as)^k,$$

where a and b are relatively prime positive integers.

a. Show that $S_{k-1} = \frac{1}{kb}\left[\sum_{r=0}^{b-1}((ab+r)^k - a^k r^k) - \sum_{m=2}^{k}\binom{k}{m} b^m S_{k-m}\right]$ for $k \geq 1$.

The convention that $\binom{k}{m} = o$ if $m > k$ is adopted.

b. Show that $S_2 = \frac{ab}{12}[3a^2b^2 + 2a^2b + 2ab^2 - a^2 - b^2 - 9ab + a + b + 2]$.

Solution by Paul S. Bruckman, Berkeley, CA

In the rs-plane, let L denote the line segment $br + as = ab$ (i.e., $r/a + s/b = 1$), with $0 \leq r \leq a$, $0 \leq s \leq b$. Also, let T denote the triangular first-quadrant region bounded by the axes and L, including points on the axes that are not on L, and excluding points on L.

Note that S_k considers only the lattice points $\{r, s\}$ that are elements of T. The only lattice points that lie on L are the points $\{a, 0\}$ and $\{0, b\}$, a consequence of the fact that $\gcd(a, b) = 1$. For brevity, we write

$$S_k = \sum_T \sum (br+as)^k,$$

where $\{r, s\}$ are lattice points of T.

We will first derive the following identity:

$$\sum_{\substack{r=0 \\ ab\le br+as<ab+b}}^{a-1}\sum_{s=0}^{b-1}(br+as)^k = \sum_{m=0}^{b-1}(ab+m)^k - \sum_{m=0}^{[(b-1)/a]}(ab+am)^k. \tag{1}$$

Proof of (1): Concentrating on the left member of (1), we see that the value of the variable $br+as$, subject to the indicated restriction, is $ab+m$, where m is an element of the set $\{0, 1, 2, \ldots, b-1\}$. Since a and b are coprime, there exist positive integers u and v such that $au-bv=1$. Assuming that $m>0$ is given and not a multiple of a, the equation $br+as=ab+m$ has solutions $\{r, s\}$ given by: $r=a-mv+at$, $s=mu-bt$, where t is an arbitrary integer. However, since we require $0\le r\le a-1$, $0\le s\le b-1$, this forces t to have a unique value, and the solutions $\{r, s\}$ are therefore unique.

On the other hand, if $m=am'$ is given, the equation $br+as=ab+am'$ implies $br\equiv 0$ (mod a), hence $a\,|\,r$. This, in turn, can only occur if $r=0$, in which case $br+as<ab$, falling outside of the range of restricted values allowed. Thus, m assumes each value in $\{0, 1, 2, \ldots, b-1\}$ exactly once, with the exception of the multiples of a, which do not occur at all (it is seen at once that m, likewise, cannot be a multiple of b). These latter values of m to be excluded therefore comprise the set $\{0, a, 2a, \ldots, [(b-1)/a]\}$. Putting these facts together establishes the identity in (1). □

Next, consider the sum $\sum_{m=0}^{k} {}_kC_m b^m S_{k-m}$, which may also be written symbolically as $(b+S)^k$, it being understood that, in such a binomial expansion, "exponents" of S are translated to subscripts. We see that, provided $k\ge 2$, such sum equals

$$(b+S)^k = S_k + kbS_{k-1} + \sum_{m=2}^{k} {}_kC_m b^m S_{k-m}. \tag{2}$$

Note that this is also true for $k=1$ if we define the sum in (2) to vanish for $k=1$. Also, however,

$$\begin{aligned}(b+S)^k &= \sum_T\sum(b+br+as)^k \\ &= \sum_{\substack{r=1 \\ br+as<ab+b}}^{a}\sum_{s=0}^{b-1}(br+as)^k = \sum_{\substack{r=0 \\ br+as<ab+b}}^{a-1}\sum_{s=0}^{b-1}(br+as)^k + \sum_{\substack{s=0 \\ as<b}}^{b-1}(ab+as)^k - \sum_{\substack{s=0 \\ as<ab+b}}^{b-1}(as)^k \\ &= S_k + U_k + \sum_{s=0}^{[(b-1)/a]}(ab+as)^k - \sum_{s=0}^{b-1}(as)^k,\end{aligned}$$

where U_k is the sum indicated in the left member of (1). Then, using the result of (1), we obtain the following:

$$(b+S)^k = S_k + \sum_{m=0}^{b-1}(ab+m)^k - \sum_{m=0}^{b-1}(am)^k.$$

Now, substituting the result in (2) and simplifying yields the result of Part a. □

We simply substitute $k=1, 2,$ or 3 in the recurrence formula just derived in order to compute S_{k-1}. Incidentally, it is to be noted that although this recurrence is not symmetric in a and b, it should be evident that the expression for S_k must be symmetric in a and b. Thus, a comparable recurrence, with a and b switched, is also true. For $k=1$, we obtain

$$bS_0 = \sum_{m=0}^{b-1} \{ab - (a-1)m\} = ab^2 - (a-1)b(b-1)/2;$$

hence, $S_0 = ab - (a-1)(b-1)/2$ or

$$S_0 = (ab + a + b - 1)/2. \tag{3}$$

Next, setting $k = 2$, we have

$$\begin{aligned} 2bS_1 &= \sum_{m=0}^{b-1} \{a^2b^2 + 2abm - (a^2-1)m^2\} - b^2 S_0 \\ &= a^2b^3 + ab^2(b-1) - (a^2-1)b(b-1)(2b-1)/6 - b^2(ab+a+b-1)/2; \end{aligned}$$

hence,

$$\begin{aligned} 12S_1 &= 6a^2b^2 + 6ab^2 - 6ab - 2a^2b^2 + 3a^2b - a^2 + 2b^2 - 3b + 1 - 3ab^2 - 3ab - 3b^2 + 3b \\ &= 4a^2b^2 + 3ab^2 + 3a^2b - a^2 - 9ab - b^2 + 1 \end{aligned}$$

or

$$S_1 = (4a^2b^2 + 3ab^2 + 3a^2b - a^2 - 9ab - b^2 + 1)/12. \tag{4}$$

Finally, setting $k = 3$, we obtain

$$\begin{aligned} 3bS_2 &= \sum_{m=0}^{b-1} \{a^3b^3 + 3a^2b^2m + 3abm^2 - (a^3-1)m^3\} - 3b^2S_1 - b^3S_0 \\ &= a^3b^4 + 3a^2b^3(b-1)/2 + 3ab^2(b-1)(2b-1)/6 - (a^3-1)b^2(b-1)^2/4 \\ &\quad - b^2(4a^2b^2 + 3ab^2 + 3a^2b - a^2 - 9ab - b^2 + 1)/4 - b^3(ab+a+b-1)/2, \end{aligned}$$

which, after simplification, reduces to the expression in Part b. □

Also solved by H.-J. Seiffert and the proposer.

Be Determinant

H-552 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 2, May 1999)

Given $m \geq 2$, let $\{U_n\}_{n=0}^{\infty}$ denote a sequence of the following form:

$$U_n = \sum_{i=1}^{m} a_i (\theta_k)^n,$$

where the a_i's and θ_i's are constants, with the θ_i's distinct, and the U_n's satisfy the initial conditions $U_n = 0$, $n = 0, 1, \ldots, m-2$; $U_{m-1} = 1$.

Part A. Prove the following formula for the U_n's:

$$U_n = \sum_{S(n-m+1,\, m)} (\theta_1)^{i_1} (\theta_2)^{i_2} \cdots (\theta_m)^{i_m}, \tag{a}$$

where

$$S(N, m) = \{(i_1, i_2, \ldots, i_m) : i_1 + i_2 + \cdots + i_m = N,\ 0 \leq i_j < N,\ j = 1, 2, \ldots, m\}. \tag{b}$$

Part B. Prove the following determinant formula for the U_n's:

$$U_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^3 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^n & (\theta_2)^n & (\theta_3)^n & \cdots & (\theta_m)^n \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_m \\ (\theta_1)^2 & (\theta_2)^2 & (\theta_3)^3 & \cdots & (\theta_m)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\theta_1)^{m-2} & (\theta_2)^{m-2} & (\theta_3)^{m-2} & \cdots & (\theta_m)^{m-2} \\ (\theta_1)^{m-1} & (\theta_2)^{m-1} & (\theta_3)^{m-1} & \cdots & (\theta_m)^{m-1} \end{vmatrix}.$$

Solution by H.-J. Seiffert, Berlin, Germany

If $V(x_1, \ldots, x_m) = \det((x_k)^{j-1})_{j,k=1,\ldots,m}$ denotes the Vandermonde determinant of m distinct numbers $x_1, \ldots, x_m$, then, as is well known,

$$V(x_1, \ldots, x_m) = \prod_{i \le j < k \le m} (x_k - x_j). \tag{1}$$

From $U_n = 0$, $n = 0, \ldots, m-2$, and $U_{m-1} = 1$, we have the following system of linear equations:

$$\begin{array}{llll} a_1 & +a_2 & +\cdots+a_m & =0, \\ a_1(\theta_1) & +a_2(\theta_2) & +\cdots+a_m(\theta_m) & =0, \\ \vdots & \vdots & \vdots & \vdots \\ a_1(\theta_1)^{m-2} & +a_2(\theta_2)^{m-2} & +\cdots+a_m(\theta_m)^{m-2} & =0, \\ a_1(\theta_1)^{m-1} & +a_2(\theta_2)^{m-1} & +\cdots+a_m(\theta_m)^{m-1} & =1. \end{array}$$

Solving by Cramer's rule and expanding the nominator determinant occurring there by the i^{th} column gives

$$a_i = (-1)^{m-i} \frac{V(\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_m)}{V(\theta_1, \ldots, \theta_m)}, \quad i = 1, \ldots, m, \tag{2}$$

where $\hat{\theta}_i$ indicates that θ_i is released. Hence, by (1),

$$a_i = \prod_{\substack{j=1 \\ j \ne i}}^{m} \frac{1}{\theta_i - \theta_j}, \quad i = 1, \ldots, m. \tag{3}$$

For sufficiently small $|x|$, we consider the generating functions

$$U(x) = \sum_{n=0}^{\infty} U_n x^n \quad \text{and} \quad W(x) = \sum_{n=0}^{\infty} \sum_{S(n-m+1,\, m)} (\theta_1)^{i_1} (\theta_2)^{i_2} \cdots (\theta_m)^{i_m} x^n.$$

Then,

$$U(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{m} a_i (\theta_i)^n x^n = \sum_{i=1}^{m} a_i \sum_{n=0}^{\infty} (\theta_i x)^n,$$

or

$$U(x) = \sum_{i=1}^{m} \frac{a_i}{1 - \theta_i x}, \tag{4}$$

and

$$W(x) = \sum_{n=0}^{\infty} \sum_{S(n-m+1,\, m)} (\theta_1 x)^{i_1} (\theta_2 x)^{i_2} \cdots (\theta_m x)^{i_m} x^{m-1} = x^{m-1} \prod_{j=1}^{m} \left(\sum_{i=0}^{\infty} (\theta_j x)^i \right),$$

or

$$W(x) = x^{m-1}\prod_{j=1}^{m}\frac{1}{1-\theta_j x}. \tag{5}$$

From Lagrange's interpolation formula, we have

$$\sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{x-\theta_j}{\theta_i-\theta_j} = 1 \text{ for all } x.$$

Replacing x by $1/x$ and then multiplying by $x^{m-1}\prod_{j=1}^{m}(1-\theta_j x)^{-1}$ yields

$$\sum_{i=1}^{m}\frac{1}{1-\theta_i x}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{1}{\theta_i-\theta_j} = x^{m-1}\prod_{j=1}^{m}\frac{1}{1-\theta_j x},$$

valid for all sufficiently small $|x|$. Now, from (3), (4), and (5), we see that $U(x) = W(x)$ for all sufficiently small $|x|$. Comparing coefficients of these generating functions gives the desired equation (a) of Part A.

Expanding the nominator determinant of the requested equation of Part B by the last row, we see that we must show:

$$U_n = \sum_{i=1}^{m}(-1)^{m-i}\frac{V(\theta_1,\ldots,\hat{\theta}_i,\ldots,\theta_m)}{V(\theta_1,\ldots,\theta_m)}(\theta_i)^n.$$

However, this holds by (2) and the definition of U_n. This completes the solution.

Also solved by the proposer.

Lotsa Terms

H-554 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 37, no. 3, August 1999)***

Let k, a, and b be positive integers with a and b relatively prime to each other, and define

$$N_k := (1+(-1)^k - L_k)^{-1} = \begin{cases}(2-L_k)^{-1} & \text{if } k \text{ is even,}\\ -L_k^{-1} & \text{if } k \text{ is odd.}\end{cases}$$

a. Show that

$$\sum_{\substack{r=0\\ br+as<ab}}^{a-1}\sum_{s=0}^{b-1}L_q(br+as) = N_{qa}N_{qb}[2+L_{q(a+b)}-L_{qa}-L_{qb}-L_{qab}+(-1)^{qa}L_{qa(b-1)} + (-1)^{qb}L_{qb(a-1)}+(-1)^{q(a+b)+1}L_{q(ab-a-b)}]+N_q[(-1)^q L_{q(ab-1)}-L_{qab}],$$

where q is a positive integer.

b. Show that

$$\sum_{\substack{r=0\\ br+as<ab}}^{a-1}\sum_{s=0}^{b-1}F_q(br+as) = N_{qa}N_{qb}[(-1)^{q(a+b)+1}F_{q(ab-a-b)}+F_{qa}+F_{qb}-F_{qab}+(-1)^{qa}F_{qa(b-1)} + (-1)^{qb}F_{qb(a-1)}-F_{q(a+b)}]+N_q[(-1)^q F_{q(ab-1)}-F_{qab}],$$

where q is a positive integer.

Solution by Paul S. Bruckman, Berkeley, CA

In the *rs*-plane, let L denote the line segment $br+as=ab$ (i.e., $r/a+s/b=1$), with $0\le r\le a$, $0\le s\le b$. Let S denote the set of lattice points included in the triangular first-quadrant region bounded by the axes and L, including points on the axes, but excluding points on L. Also, let T denote the triangular "mirror-image" of S about L, and let $R=S\cup T\cup(0,b)\cup(a,0)$. Thus, R is the set of all lattice points included in the rectangular region with $0\le r\le a$, $0\le s\le b$. Since $\gcd(a,b)=1$, we see that the only lattice points of $R\cap L$ are at the end-points of L, namely, at $(0,b)$ and $(a,0)$; moreover, these end-points are neither in S nor in T.

For brevity, write $N=br+as$, and $N\in S$ to mean $(r,s)\in S$, with similar notation for T and for R. Make the following definitions, valid for arbitrary $x\neq 1$:

$$S(x)=\sum_{N\in S}x^N,\quad T(x)=\sum_{N\in T}x^N,\quad R(x)=\sum_{N\in R}x^N. \tag{1}$$

We see that

$$\begin{aligned}R(x)&=(1+x^a+x^{2a}+\cdots+x^{ba})(1+x^b+x^{2b}+\cdots+x^{ab})\\&=(1-x^{(b+1)a})(1-x^{(a+1)b})/\{(1+x^a)(1-x^b)\}.\end{aligned}$$

We also see that $R(x)=S(x)+T(x)+2x^{ab}$ and, moreover, that $T(x)=\sum_{N\in S}x^{2ab-N}=x^{2ab}S(x^{-1})$. This yields the following symmetrical relation (upon division by x^{ab}):

$$x^{-ab}S(x)+x^{ab}S(x^{-1})+2=x^{-ab}R(x)=U(x)\ \text{(say)}. \tag{2}$$

As we may verify, $U(x)=x^{-ab}(1-x^{(b+1)a})(1-x^{(a+1)b})/\{(1-x^a)(1-x^b)\}=U(x^{-1})$.

We also observe, due to the fact that $\gcd(a,b)=1$, that each value of N occurring as an exponent in the sum $S(x)$ occurs but once.

We may evaluate the sum $S(x)$ by means of certain manipulations. Thus,

$$\begin{aligned}S(x)&=\sum_{N\in S}x^N=\sum_{\substack{r=0\\ br+as<ab}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}=\sum_{\substack{r=1\\ b(r-1)+as<ab}}^{a}\sum_{s=0}^{b-1}x^{b(r-1)+as}\\&=x^{-b}\sum_{\substack{r=0\\ br+as<b(a+1)}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}-x^{-b}\sum_{s=0}^{b-1}x^{as}+x^{-b}\sum_{s=0}^{[(b-1)/a]}x^{ba+as};\end{aligned}$$

assuming that $a\ge b$, then

$$\begin{aligned}x^bS(x)&=\sum_{\substack{r=0\\ br+as<ab}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}+\sum_{\substack{r=0\\ ab<br+as<ab+b}}^{a-1}\sum_{s=0}^{b-1}x^{br+as}-(x^{ab}-1)/(x^a-1)+x^{ab}\\&=S(x)+\sum_{k=ab+1}^{ab+b-1}x^k-(x^{ab}-1)/(x^a-1)+x^{ab}\\&=S(x)+(x^{ab+b}-x^{ab+1})/(x-1)-(x^{ab}-1)/(x^a-1)+x^{ab}.\end{aligned}$$

After simplification, we obtain the symmetric expression (valid also if $a\le b$):

$$S(x)=(1-x^{ab})/\{(1-x^a)(1-x^b)\}-x^{ab}/(1-x). \tag{3}$$

Now the sums given in the problem are seen to equal the following expressions:

$$\sum_{N\in S} L_{qN} = S(\alpha^q)+S(\beta^q);\quad \sum_{N\in S} F_{qN} = 5^{-1/2}\{S(\alpha^q)-S(\beta^q)\}. \tag{4}$$

Therefore, it remains to show that the expressions in (4) may be simplified to the expressions given in the statement of the problem.

Note that $(\alpha^{qk}-1)(\beta^{qk}-1)=(-1)^{qk}+1-L_{qk}=1/N_{qk}$. Then

$$\begin{aligned}S(\alpha^q) &= (1-\alpha^{qab})/\{(1-\alpha^{qa})(1-\alpha^{qb})\}-\alpha^{qab}/(1-\alpha^q)\\ &= N_{qa}N_{qb}(1-\alpha^{qab})(1-\beta^{qa})(1-\beta^{qb})-N_q\alpha^{qab}(1-\beta^q)\\ &= N_{qa}N_{qb}(1-\alpha^{qab})(1-\beta^{qa}-\beta^{qb}+\beta^{q(a+b)})-N_q\alpha^{qab}(1-\beta^q)\end{aligned}$$

or

$$\begin{aligned}S(\alpha^q) = N_{qa}N_{qb}\{&1-\beta^{qa}-\beta^{qb}+\beta^{q(a+b)}-\alpha^{qab}+(-1)^{qa}\alpha^{qab-qa}\\ &+(-1)^{qb}\alpha^{qab-qb}-(-1)^{qa+qb}\alpha^{qab-qa-qb}\}-N_q(\alpha^{qab}-(-1)^q\alpha^{qab-q}).\end{aligned} \tag{5}$$

Likewise,

$$\begin{aligned}S(\beta^q) = N_{qa}N_{qb}\{&1-\alpha^{qa}-\alpha^{qb}+\alpha^{q(a+b)}-\beta^{qab}+(-1)^{qa}\beta^{qab-qa}\\ &+(-1)^{qb}\beta^{qab-qb}-(-1)^{qa+qb}\beta^{qab-qa-qb}\}-N_q(\beta^{qab}-(-1)^q\beta^{qab-q}).\end{aligned} \tag{6}$$

Now, adding the expressions in (5) and (6) (and using (4)), we obtain

$$\begin{aligned}\sum_{N\in S} L_{qN} = N_{qa}N_{qb}\{&2-L_{qa}-L_{qb}+L_{q(a+b)}-L_{qab}+(-1)^{qa}L_{qa(b-1)}\\ &+(-1)^{qb}L_{qb(a-1)}-(-1)^{q(a+b)}L_{q(ab-a-b)}\}-N_q\{L_{qab}-(-1)^q L_{q(ab-1)}\},\end{aligned}$$

which is seen to be equivalent to the result of Part a.

Subtracting the two expressions in (5) and (6) and dividing by $5^{1/2}$ yields

$$\begin{aligned}\sum_{N\in S} F_{qN} = N_{qa}N_{qb}\{&F_{qa}+F_{qb}-F_{q(a+b)}-F_{qab}+(-1)^{qa}F_{qa(b-1)}\\ &+(-1)^{qb}F_{qb(a-1)}-(-1)^{q(a+b)}F_{q(ab-a-b)}\}-N_q\{F_{qab}-(-1)^q F_{q(ab-1)}\},\end{aligned}$$

which is seen to be equivalent to the result of Part b, except for a small error.

We note that the statement of the problem in the August 1999 issue of this journal contains a typographical error: the term in Part b that reads "$+F_{qb(a-1)}$" should read "$+(-1)^{qb}F_{qb(a-1)}$". The problem is stated correctly above. □

Also solved by H.-J. Seiffert and the proposer.

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