SUMMATION OF RECIPROCALS WHICH INVOLVE PRODUCTS OF TERMS FROM GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

We consider the sequence $\{W_n\}$ defined, for all integers n, by

$$W_n = pW_{n-1} + W_{n-2}, \quad W_0 = a, \quad W_1 = b. \tag{1.1}$$

Here a, b, and p are real numbers with p strictly positive. Write $\Delta = p^2 + 4$. Then it is well known [5] that

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.2}$$

where $\alpha = (p+\sqrt{\Delta})/2$, $\beta = (p-\sqrt{\Delta})/2$, $A = b - a\beta$, and $B = b - a\alpha$. As in [5], we put $e_W = AB = b^2 - pab - a^2$.

We define a companion sequence $\{\overline{W}_n\}$ of $\{W_n\}$ by

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.3}$$

Aspects of this sequence have been treated, for example, in [6] and [7]. In the first of these references $\{W_n\}$ and $\{\overline{W}_n\}$ are denoted by $\{H_n\}$ and $\{K_n\}$, respectively.

For $(W_0, W_1) = (0, 1)$ we write $\{W_n\} = \{U_n\}$, and for $(W_0, W_1) = (2, p)$ we write $\{W_n\} = \{V_n\}$. The sequences $\{U_n\}$ and $\{V_n\}$ are generalizations of the Fibonacci and Lucas sequences, respectively. From (1.2) and (1.3), we see that $\overline{U}_n = V_n$ and $\overline{V}_n = \Delta U_n$. It is clear that $e_U = 1$ and $e_V = -\Delta = -(\alpha-\beta)^2$.

The purpose of this paper is to investigate certain infinite sums. In Section 3 we investigate the sum

$$S_{k,m} = \sum_{n=1}^{\infty} \frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}, \tag{1.4}$$

and in Section 4 we investigate the sum

$$T_{k,m} = \sum_{n=1}^{\infty} \frac{(-1)^n}{W_{kn}W_{k(n+m)}W_{k(n+2m)}W_{k(n+3m)}}, \tag{1.5}$$

where k and m are taken to be odd positive integers.

Now since $p > 0$, then $\alpha > 1$ and $\alpha > |\beta|$, so that

$$W_n \cong \frac{A}{\alpha - \beta}\alpha^n \quad \text{and} \quad \overline{W}_n \cong A\alpha^n. \tag{1.6}$$

Hence, assuming that a and b are chosen so that no denominator vanishes, we see from the ratio test that $S_{k,m}$ and $T_{k,m}$ are absolutely convergent.

2. PRELIMINARY RESULTS

We require the following, in which k and m are taken to be odd integers.

$$W_{n+k}+W_{n-k}=\overline{W}_n U_k, \tag{2.1}$$

$$W_{n+k}-W_{n-k}=W_n V_k, \tag{2.2}$$

$$\beta W_n+W_{n-1}=B\beta^n, \tag{2.3}$$

$$\alpha^m W_{n+m}+W_n=A\alpha^{n+m}U_m, \tag{2.4}$$

$$W_{k(n+m)}W_{k(n+2m)}-W_{kn}W_{k(n+3m)}=e_W(-1)^n U_{km}U_{2km}, \tag{2.5}$$

$$\sum_{n=n_1}^{n_2}\frac{1}{\alpha^{kn}W_{kn}}=\frac{1}{B}\sum_{n=n_1}^{n_2}(-1)^n\frac{W_{kn-1}}{W_{kn}},\quad n_2-n_1 \text{ odd}, \tag{2.6}$$

$$\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+m)}W_{k(n+m)}}=A\frac{U_{km}}{W_{kn}W_{k(n+m)}}. \tag{2.7}$$

Identities (2.1)-(2.5) are readily proved with the use of (1.2) and (1.3). Now, since k is odd, then $\alpha^{-kn}=(-1)^{kn}\beta^{kn}=(-1)^n\beta^{kn}$. Hence,

$$\begin{aligned}\sum_{n=n_1}^{n_2}\frac{1}{\alpha^{kn}W_{kn}}&=\sum_{n=n_1}^{n_2}\frac{(-1)^n\beta^{kn}}{W_{kn}}\\&=\frac{1}{B}\sum_{n=n_1}^{n_2}\frac{(-1)^n(\beta W_{kn}+W_{kn-1})}{W_{kn}},\quad\text{by (2.3)},\\&=\frac{1}{B}\sum_{n=n_1}^{n_2}\left((-1)^n\beta+(-1)^n\frac{W_{kn-1}}{W_{kn}}\right),\end{aligned}$$

and since n_2-n_1+1 is even, this yields (2.6). Identity (2.7) is readily established with the use of (2.4).

We also require the following theorem, which follows immediately from (2.7).

Theorem 1: If k and m are odd positive integers, then

$$AU_{km}\sum_{n=1}^{\infty}\frac{1}{W_{kn}W_{k(n+m)}}=2\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}-\sum_{n=1}^{m}\frac{1}{\alpha^{kn}W_{kn}}. \tag{2.8}$$

Since $\alpha>1$ and $\alpha>|\beta|$, it follows from the ratio test that the infinite sums in (2.8) are absolutely convergent. For similar infinite sums in which the denominator consists of products of two terms from the sequence $\{W_n\}$, see [2].

3. THE SUM $S_{k,m}$

The first of two theorems in this section is

Theorem 2: If k and m are odd positive integers, then

$$AU_{km}^2 S_{k,m}=4\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}-\sum_{n=1}^{2m}\frac{1}{\alpha^{kn}W_{kn}}-2\sum_{n=1}^{m}\frac{1}{\alpha^{kn}W_{kn}}. \tag{3.1}$$

Proof: Consider the expression

$$\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+m)}W_{k(n+m)}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}. \tag{3.2}$$

Using (2.7), we can write this as

$$\frac{AU_{km}}{W_{kn}W_{k(n+m)}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}, \tag{3.3}$$

or as

$$\frac{1}{\alpha^{kn}W_{kn}}+\frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}}. \tag{3.4}$$

Now

$$\begin{aligned}\frac{AU_{km}}{W_{kn}W_{k(n+m)}}+\frac{AU_{km}}{W_{k(n+m)}W_{k(n+2m)}}&=\frac{AU_{km}}{W_{k(n+m)}}\left[\frac{1}{W_{kn}}+\frac{1}{W_{k(n+2m)}}\right]\\&=\frac{AU_{km}}{W_{k(n+m)}}\cdot\frac{W_{k(n+2m)}+W_{kn}}{W_{kn}W_{k(n+2m)}}\\&=AU_{km}^2\cdot\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}},\quad\text{by (2.1).}\end{aligned} \tag{3.5}$$

But, from (3.2)-(3.4), we then have

$$\begin{aligned}&2\left[\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+m)}W_{k(n+m)}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}\right]\\&=\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}+AU_{km}^2\cdot\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}},\end{aligned}$$

so that

$$AU_{km}^2\cdot\frac{\overline{W}_{k(n+m)}}{W_{kn}W_{k(n+m)}W_{k(n+2m)}}=\frac{1}{\alpha^{kn}W_{kn}}+\frac{1}{\alpha^{k(n+2m)}W_{k(n+2m)}}+\frac{2}{\alpha^{k(n+m)}W_{k(n+m)}}.$$

Now, summing both sides, we obtain (3.1). □

Our next theorem expresses $S_{k,m}$ in terms of $S_{k,1}$.

Theorem 3: Let k and m be odd positive integers with $m>1$. Then

$$AU_{km}^2S_{k,m}=AU_k^2S_{k,1}-\frac{1}{B}\left[\sum_{n=3}^{2m}(-1)^n\frac{W_{kn-1}}{W_{kn}}+2\sum_{n=2}^{m}(-1)^n\frac{W_{kn-1}}{W_{kn}}\right]. \tag{3.6}$$

Proof: From (3.1), we have

$$AU_k^2S_{k,1}=4\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}-\sum_{n=1}^{2}\frac{1}{\alpha^{kn}W_{kn}}-\frac{2}{\alpha^kW_k}. \tag{3.7}$$

In (3.7), we solve for

$$4\sum_{n=1}^{\infty}\frac{1}{\alpha^{kn}W_{kn}}$$

and substitute in (3.1) to obtain

$$AU_{km}^2 S_{k,m} = AU_k^2 S_{k,1} - \sum_{n=3}^{2m} \frac{1}{\alpha^{kn} W_{kn}} - 2\sum_{n=2}^{m} \frac{1}{\alpha^{kn} W_{kn}}.$$

From this, we arrive at (3.6) by using (2.6). □

For an application of Theorem 3, take $k = 1$ and $m = 3$. Then, with $W_n = F_n$ and $W_n = L_n$, (3.6) becomes, respectively,

$$\sum_{n=1}^{\infty} \frac{L_{n+3}}{F_n F_{n+3} F_{n+6}} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{L_{n+1}}{F_n F_{n+1} F_{n+2}} - \frac{143}{480}, \tag{3.8}$$

and

$$\sum_{n=1}^{\infty} \frac{F_{n+3}}{L_n L_{n+3} L_{n+6}} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{F_{n+1}}{L_n L_{n+1} L_{n+2}} - \frac{115}{11088}. \tag{3.9}$$

4. THE SUM $T_{k,m}$

We denote the infinite sum on the left side of (2.8) by

$$t_{k,m} = \sum_{n=1}^{\infty} \frac{1}{W_{kn} W_{k(n+m)}}.$$

Then, from (2.8), we see that

$$\begin{cases} AU_{3km} t_{k,3m} = 2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} - \sum_{n=1}^{3m} \frac{1}{\alpha^{kn} W_{kn}}, \\ AU_{km} t_{k,m} = 2\sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} - \sum_{n=1}^{m} \frac{1}{\alpha^{kn} W_{kn}}. \end{cases}$$

Next, we solve for $t_{k,3m}$ and $t_{k,m}$ and consider their difference. Then, making use of (2.2) to factor $U_{3km} - U_{km}$, and noting that $U_{2n} = U_n V_n$, we obtain

$$A(t_{k,3m} - t_{k,m}) = \frac{-2V_{km}^2}{U_{3km}} \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} + \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{\alpha^{kn} W_{kn}} - \frac{1}{U_{3km}} \sum_{n=1}^{3m} \frac{1}{\alpha^{kn} W_{kn}}. \tag{4.1}$$

Our main result concerning $T_{k,m}$ can now be given in the following theorem.

Theorem 4: Let k and m be odd positive integers. Then

$$\begin{aligned} e_W A U_{km} U_{2km} T_{k,m} = {} & \frac{-2V_{km}^2}{U_{3km}} \sum_{n=1}^{\infty} \frac{1}{\alpha^{kn} W_{kn}} + \frac{1}{U_{km}} \sum_{n=1}^{m} \frac{1}{\alpha^{kn} W_{kn}} \\ & - \frac{1}{U_{3km}} \sum_{n=1}^{3m} \frac{1}{\alpha^{kn} W_{kn}} + A \sum_{n=1}^{m} \frac{1}{W_{kn} W_{k(n+m)}}. \end{aligned} \tag{4.2}$$

Proof: Using (2.5), we see that

$$\frac{e_W (-1)^n U_{km} U_{2km}}{W_{kn} W_{k(n+m)} W_{k(n+2m)} W_{k(n+3m)}} = \frac{1}{W_{kn} W_{k(n+3m)}} - \frac{1}{W_{k(n+m)} W_{k(n+2m)}}.$$

If we now sum both sides, we obtain

$$e_W U_{km} U_{2km} T_{k,m} = t_{k,3m} - \left[t_{k,m} - \sum_{n=1}^{m} \frac{1}{W_{kn} W_{k(n+m)}} \right],$$

and (4.2) follows from (4.1). □

We mention that $T_{k,m}$ can be expressed in terms of $T_{k,1}$. We simply write down (4.2) for the case $m = 1$, solve for $\sum_{n=1}^{\infty}(1/\alpha^{kn}W_{kn})$, and then substitute in (4.2). Since the result is rather lengthy, we do not give it here.

As can be seen from Theorems 2 and 4, $S_{k,m}$ and $T_{k,m}$ can be expressed in terms of the infinite sum $\sum_{n=1}^{\infty}(1/\alpha^{kn}W_{kn})$ together with certain finite sums. If we consider specializations $W_n = U_n$ or $W_n = V_n$, this infinite sum can be expressed in terms of the Lambert series, which is defined as

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}, \quad |x| < 1. \text{ In this regard, see [1].}$$

Remark: For the sake of definiteness, we have assumed throughout this paper that $p > 0$, so that $\sum_{n=1}^{\infty}(1/\alpha^{kn}W_{kn})$ is absolutely convergent. However, we can immediately write down parallel results for $p < 0$. For then we see that $\beta < -1$ and $|\beta| > |\alpha|$, so that $W_n \cong (-B/(\alpha - \beta))\beta^n$ and $\overline{W}_n \cong B\beta^n$. It follows from the ratio test that $\sum_{n=1}^{\infty}(1/\beta^{kn}W_{kn})$ is absolutely convergent. We then obtain counterparts of Theorems 1 through 4 if in each theorem we replace $\alpha(\beta)$ by $\beta(\alpha)$ and $A(B)$ by $B(A)$. Indeed, these substitutions are valid in (2.3), (2.4), (2.6), and (2.7), regardless of the sign of p.

Finally, two early references that touch on a wide variety of infinite sums in which the denominators of the summands contain products of Fibonacci and Lucas numbers are [3] and [4].

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