CONVERGENT ∞-GENERALIZED FIBONACCI SEQUENCES

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1. INTRODUCTION

In [4], the authors have defined ∞-generalized Fibonacci sequences, which are defined by recurrence formulas involving infinitely many terms and which are generalizations of weighted r-generalized Fibonacci sequences with r finite as defined in [1]. In this paper we study the convergence property of such sequences and their associated series.

Let us first recall the definition of ∞-generalized Fibonacci sequences. Take an infinite sequence $\{a_i\}_{i=0}^{\infty}$ of complex numbers. We set $h(z) = \sum_{i=0}^{\infty} a_i z^i$ for $z \in \mathbf{C}$ and $u(x) = \sum_{i=1}^{\infty} |a_i| x^i$ for $x \in \mathbf{R}$. Let R denote the radius of convergence of the power series h, which coincides with that of u. We assume the following:

$$0 < R \leq \infty. \tag{1.1}$$

Let X be the set of the sequences $\{x_i\}_{i=0}^{\infty}$ of complex numbers such that there exist $C > 0$ and T with $0 < T < R$ satisfying $|x_i| \leq CT^i$ for all i. Note that X is an infinite dimensional vector space over $\mathbf{C}$, which will be the set of initial sequences for ∞-generalized Fibonacci sequences associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Define $f : X \to \mathbf{C}$ by $f(x_0, x_1, \ldots) = \sum_{i=0}^{\infty} a_i x_i$. Since the series $\sum_{i=0}^{\infty} a_i CT^i$ converges absolutely, the series defining f also converges absolutely. Then, for a sequence $\{y_0, y_{-1}, y_{-2}, \ldots\} \in X$, we define the sequence $\{y_1, y_2, y_3, \ldots\}$ by

$$y_n = f(y_{n-1}, y_{n-2}, y_{n-3}, \ldots) = \sum_{i=1}^{\infty} a_{i-1} y_{n-i} \quad (n = 1, 2, 3, \ldots),$$

which is well defined as is shown in [4]. The sequence $\{y_i\}_{i \in \mathbf{Z}}$ is called an *∞-generalized Fibonacci sequence* associated with the weight sequence $\{a_i\}_{i=0}^{\infty}$. Note that if there exists an integer $r \geq 1$ such that $a_i = 0$ for all $i \geq r$, then the sequence $\{a_i\}_{i=0}^{\infty}$ satisfies condition (1.1) and the above definition coincides with that of weighted r-generalized Fibonacci sequences with r finite (see [1]).

Note that, as far as the authors know, this is a new generalization of the usual Fibonacci sequences and almost nothing has been known about such sequences until now, except those results obtained in [4]. For example, the following questions naturally arise.

(Q1) Are they combinations of geometric progressions, as in the finite case?
(Q2) Are they asymptotically geometric?
(Q3) Do they converge to limits?
(Q4) Do their sums converge to limits?
(Q5) Is it possible to express the n^{th} term of such a sequence as a function of n in some nice way?

We briefly recall the results obtained in [4], which are fundamental for the present paper and which give an answer to (Q2) above.

Lemma 1.2 ([4], Lemma 2.3): ***(1)*** Suppose that each a_i is a nonnegative real number and that there exists an S with $0 < S < R$ satisfying

$$a_0 > S^{-1} - u(S) \text{ (or, equivalently, } Sh(S) > 1). \tag{1.2.1}$$

Then there exists a unique $q \in \mathbf{R}$ such that $q > S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$, and $f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$.

(2) Suppose there exists an S with $0 < S < R$ satisfying

$$|a_0| > S^{-1} + u(S). \tag{1.2.2}$$

Then there exists a unique $q \in \mathbf{C}$ such that $|q| > S^{-1}$, $\{q^{-(i+1)}\}_{i=0}^{\infty} \in X$, and $f(q^{-1}, q^{-2}, q^{-3}, \dots) = 1$.

Note that, in the finite case [1], the above q corresponds to the root of the characteristic polynomial of maximal modulus. As has been seen in [4], the existence of such a q plays an important role in studying the asymptotic behavior of ∞-generalized Fibonacci sequences [see (Q2) above] and hence in exploring questions (Q3) and (Q4) above (see §5). More precisely, the following has been proved in [4].

Theorem 1.3 ([4], Theorem 3.10): Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of complex numbers that satisfies (1.1) and admits an S with $0 < S < R$ satisfying (1.2.1) or (1.2.2), and

$$S^2 u'(S) < 1. \tag{1.3.1}$$

Then $\lim_{n\to\infty} y_n / q^n$ exists and is equal to

$$\frac{\sum_{m=0}^{\infty} b_m q^m y_{-m}}{\sum_{m=0}^{\infty} b_m} \quad \text{with} \quad b_m = \sum_{i=m}^{\infty} \frac{a_i}{q^{i+1}}.$$

In the following, *we always assume that the conditions of Theorem 1.3 are satisfied.* These conditions demand that the modulus of the leading weight coefficient a_0 should be sufficiently large (see also §5). There are many sequences that satisfy these conditions. For example, take an arbitrary holomorphic function $h_1(z)$ defined in a neighborhood of zero. Then the sequence appearing as the coefficients of the power series expansion of the holomorphic function $h(z) = h_1(z) + a$ at $z = 0$ satisfies the above conditions for all $a \in \mathbf{C}$ with sufficiently large modulus $|a|$.

In this paper we consider questions (Q3) and (Q4) mentioned above and prove the following results, which give answers to the questions in certain situations.

Theorem 1.4: Suppose that each a_i is a nonnegative real number and that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. The following three are equivalent.
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ does not converge.
   (b) $\sum_{i=0}^{\infty} a_i > 1$.
   (c) $q > 1$.

2. The following three are equivalent.
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ converges to a nonzero real number.
   (b) $\sum_{i=0}^{\infty} a_i = 1$.
   (c) $q = 1$.

Furthermore, in case 2(a), we have

$$\lim_{n\to\infty} y_n = \frac{\sum_{m=0}^{\infty} b_m y_{-m}}{\sum_{m=0}^{\infty} b_m} \quad \text{with} \quad b_m = \sum_{i=m}^{\infty} a_i .$$

3. The following three are equivalent.
   (a) The sequence $\{y_n\}_{n=1}^{\infty}$ converges to zero.
   (b) $\sum_{i=0}^{\infty} a_i < 1$.
   (c) $q < 1$.

Theorem 1.5: Suppose that each a_i is a nonnegative real number and that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. The following three are equivalent.
   (a) The series $\sum_{n=1}^{\infty} y_n$ does not converge.
   (b) $\sum_{i=0}^{\infty} a_i \geq 1$.
   (c) $q \geq 1$.

2. The following three are equivalent.
   (a) The series $\sum_{n=1}^{\infty} y_n$ converges.
   (b) $\sum_{i=0}^{\infty} a_i < 1$.
   (c) $q < 1$.

Furthermore, in case 2(a), we have

$$\sum_{n=1}^{\infty} y_n = \frac{\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} a_i y_{j-i}}{1-\sum_{i=0}^{\infty} a_i} = \frac{\sum_{m=0}^{\infty}\left(\sum_{i=m}^{\infty} a_i\right) y_{-m}}{1-\sum_{i=0}^{\infty} a_i}.$$

When a_i are general complex numbers, we have the following theorem.

Theorem 1.6: Suppose that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. We have the implications ***(b)*** $\Rightarrow$ ***(c)*** $\Leftrightarrow$ ***(a)*** among the following:
   (a) The sequence $\{|y_n|\}_{n=1}^{\infty}$ converges to ∞.
   (b) $|a_0| - \sum_{i=1}^{\infty} |a_i| > 1$.
   (c) $|q| > 1$.

2. The following two are equivalent.

(a) The sequence $\{|y_n|\}_{n=1}^{\infty}$ converges to a nonzero real number.

(b) $|q| = 1$.

Furthermore, in case 2(a), we have

$$\lim_{n\to\infty} |y_n| = \left| \frac{\sum_{m=0}^{\infty} b_m q^m y_{-m}}{\sum_{m=0}^{\infty} b_m} \right|.$$

3. We have the implications $(b) \Rightarrow (c) \Leftrightarrow (a)$ among the following:

(a) The sequence $\{y_n\}_{n=1}^{\infty}$ converges to zero.

(b) $\sum_{i=0}^{\infty} |a_i| < 1$.

(c) $|q| < 1$.

Theorem 1.7: Suppose that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$.

1. The following two are equivalent.

(a) The series $\sum_{n=1}^{\infty} |y_n|$ does not converge.

(b) $|q| \geq 1$.

2. The following two are equivalent.

(a) The series $\sum_{n=1}^{\infty} |y_n|$ converges.

(b) $|q| < 1$.

Furthermore, in case 2(a), we have

$$\sum_{n=1}^{\infty} y_n = \frac{\sum_{j=0}^{\infty} \sum_{i=j}^{\infty} a_i y_{j-i}}{1 - \sum_{i=0}^{\infty} a_i} = \frac{\sum_{m=0}^{\infty} \left(\sum_{i=m}^{\infty} a_i \right) y_{-m}}{1 - \sum_{i=0}^{\infty} a_i}.$$

Note that the above results generalize some of the results of Gerdes [2], [3], concerning weighted r-generalized Fibonacci sequences with $r = 2$ and 3.

The paper is organized as follows: In §2 we prove the convergence result, Theorem 1.4, for the nonnegative real case. In §3 we prove the convergence result, Theorem 1.6, for the general case. In §4 we give an explicit formula for the generating functions of ∞-generalized Fibonacci sequences that generalize a result of Raphael [5], and prove the convergence results for the series, i.e., Theorems 1.5 and 1.7. Finally, in §5 we give some remarks concerning questions (Q1)-(Q5) mentioned above.

2. CONVERGENCE OF SEQUENCE—NONNEGATIVE REAL CASE

In this section, we prove Theorem 1.4.

Lemma 2.1: If $K = \sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$ and $|q| > 1$, then $\lim_{n\to\infty} |y_n| = \infty$.

Proof: By Theorem 1.3, there exists an integer N such that, for all $n \geq N$, we have

$$|(y_n / q^n) - K| < |K|/2.$$

In particular, we have

$$|y_n| \geq |K||q^n| - |K - (y_n / q^n)||q^n| > |K||q|^n/2$$

for all $n \geq N$. Then the result is obvious. □

Lemma 2.2: If $|q| < 1$, then $\lim_{n\to\infty}|y_n| = 0$.

Proof: By Theorem 1.3, there exists an integer N such that, for all $n > N$, we have

$$|(y_n / q^n) - K| < 1.$$

Then we have

$$|y_n / q^n| \leq |(y_n / q^n) - K| + |K| < |K| + 1;$$

hence, $|y_n| < (|K| + 1)|q|^n$ for all $n \geq N$. Then the result is obvious. □

Lemma 2.3: Suppose that each a_i is a nonnegative real number.

(1) If $\sum_{i=0}^{\infty} a_i > 1$, then $q > 1$.

(2) If $\sum_{i=0}^{\infty} a_i = 1$, then $q = 1$.

(3) If $\sum_{i=0}^{\infty} a_i < 1$, then $q < 1$.

Proof: Let $\varphi : [0, R) \to \mathbf{R}$ be the function defined by $\varphi(x) = xh(x)$, which is strictly increasing. Note that $\varphi(x) = 1$ if and only if $f(x, x^2, x^3, \ldots) = 1$.

(1) When $S \leq 1$, we have nothing to prove, since $q > S^{-1}$. When $S > 1$, we have $1 < S < R$ by our assumption and

$$\varphi(1) = h(1) = \sum_{i=0}^{\infty} a_i > 1 = \varphi(q^{-1})$$

by Lemma 1.2. Thus, we have $1 > q^{-1}$, which implies that $q > 1$.

(2) Since $\sum_{i=0}^{\infty} a_i$ converges, we have $R \geq 1$. If $R > 1$, then $x = 1$ is the unique solution of the equation $\varphi(x) = 1$ on the interval $[0, R)$. Thus, $q^{-1} = 1$ by Lemma 1.2. If $R = 1$, then φ can be extended to a strictly increasing function on $[0, R]$ with $\varphi(1) = 1$. This contradicts the assumption that $\varphi(S) > 1$ for some $S \in [0, R)$.

(3) Since $\sum_{i=0}^{\infty} a_i$ converges, we have $R \geq 1$. By the same argument as in (2), we have $R > 1$. Then we have $\varphi(1) < 1 = \varphi(q^{-1})$ by Lemma 1.2. Thus, we have $1 < q^{-1}$, which implies that $q < 1$. □

Theorem 1.4 follows from the above lemmas together with Theorem 1.3.

The condition that $\sum_{m=0}^{\infty} b_m q^m y_{-m} \neq 0$ is satisfied, for example, for $y_i = g_i$, where $\{g_i\}_{i \in \mathbf{Z}}$ is the sequence as defined in [4, §3]. Thus, we have the following corollary.

Corollary 2.4: Let $\{g_i\}_{i=1}^{\infty}$ be the ∞-generalized Fibonacci sequence associated with the initial sequence $\{g_{-i}\}_{i=0}^{\infty}$, where $g_0 = 1$ and $g_{-i} = 0$ for $i \geq 1$. If all a_i are nonnegative real numbers and if $\sum_{i=0}^{\infty} a_i = 1$, then the sequence $\{g_i\}_{i=1}^{\infty}$ converges to

$$\left(\sum_{m=0}^{\infty} b_m\right)^{-1} = \left(\sum_{i=0}^{\infty} (i+1) a_i\right)^{-1} = \frac{1}{h(1) + h'(1)}.$$

3. CONVERGENCE OF SEQUENCE—GENERAL CASE

In this section we prove Theorem 1.6.

Lemma 3.1: If $|a_0| - \Sigma_{i=1}^{\infty}|a_i| > 1$, then $|q| > 1$.

Proof: When $S \le 1$, we have $|q| > 1$ by Lemma 1.2. When $S > 1$, we have $1 < S < R$. Consider the function $t:[0, R) \to \mathbf{R}$ defined by $t(x) = x^2 u'(x)$, which is strictly increasing. Then we have $t(1) < t(S) < 1$ by condition (1.3.1). Furthermore, by our assumption, we have $|a_0| > 1 + u(1)$. Hence, in (1.2.2) and (1.3.1), we may assume that $S = 1$. Thus, we have $|q| > 1$ by Lemma 1.2. This completes the proof. □

Lemma 3.2: If $\Sigma_{i=0}^{\infty}|a_i| < 1$, then $|q| < 1$.

Proof: We have

$$\sum_{i=0}^{\infty}|a_i| < 1 = f(q^{-1}, q^{-2}, q^{-3}, \dots) = \left|\sum_{i=0}^{\infty} a_i q^{-(i+1)}\right| \le \sum_{i=0}^{\infty}|a_i||q^{-1}|^{i+1}.$$

Thus, we have $|q^{-1}| > 1$. This completes the proof. □

Theorem 1.6 follows from the above lemmas together with the lemmas in §2 and Theorem 1.3.

Corollary 3.3: Let $\{g_i\}_{i=1}^{\infty}$ be the ∞-generalized Fibonacci sequence associated with the initial sequence $\{g_{-i}\}_{i=0}^{\infty}$, where $g_0 = 1$ and $g_{-i} = 0$ for $i \ge 1$. Then the sequence $\{g_i\}_{i=1}^{\infty}$ converges to a nonzero complex number if and only if $|q| = 1$.

4. GENERATING FUNCTION AND CONVERGENT SERIES

First, we prove the following formula for the generating function of ∞-generalized Fibonacci sequences, which generalizes a result of Raphael [5].

Theorem 4.1: Suppose that the sequence $\{a_i\}_{i=0}^{\infty}$ satisfies the condition in Lemma 1.2. Then the generating function of the sequence $\{y_i\}_{i=1}^{\infty}$ is equal to

$$\sum_{i=0}^{\infty} y_{i+1} z^i = \frac{k(z)}{1 - zh(z)},$$

where $h(z) = \Sigma_{i=0}^{\infty} a_i z^i$ and

$$k(z) = \sum_{j=0}^{\infty}\left(\sum_{i=j}^{\infty} a_i y_{j-i}\right) z^j.$$

More precisely, the above equality holds for all $z \in \mathbf{C}$ with $|z| < |q|^{-1}$.

Proof: First, consider the power series $k(z)$. Let the radius of convergence of k be denoted by R'. Then we have

$$R' = \left(\limsup_{j\to\infty} \sqrt[j]{\left|\sum_{i=j}^{\infty} a_i y_{j-i}\right|}\right)^{-1}.$$

Since the sequence $\{y_0, y_{-1}, y_{-2}, \dots\}$ is an element of X, there exist $C > 0$ and T with $0 < T < R$ such that $y_{-i} \le CT^i$ for all $i \ge 0$. Thus, we have

$$\limsup_{j\to\infty} \sqrt[j]{\left|\sum_{i=j}^{\infty} a_i y_{j-i}\right|} \le \limsup_{j\to\infty} \sqrt[j]{\sum_{i=j}^{\infty} |a_i| CT^{i-j}} = \limsup_{j\to\infty} \sqrt[j]{\frac{C}{T^j}\sum_{i=j}^{\infty} |a_i| T^i}$$
$$\le \limsup_{j\to\infty} \frac{\sqrt[j]{C}}{T}\sqrt[j]{u(T)} = \frac{1}{T}.$$

Thus, we have $R' \ge T$. Since we can choose T as close to R as we want, we have $R' \ge R$. Thus, in particular, for $z \in \mathbf{C}$ with $|z| < R$, the series $k(z)$ converges absolutely.

Therefore, for z with $|z| < R$, we have

$$\begin{aligned}(1-zh(z))\left(\sum_{i=0}^{\infty} y_{i+1}z^i\right) &= \left(1-\sum_{i=0}^{\infty} a_i z^{i+1}\right)\left(\sum_{i=0}^{\infty} y_{i+1}z^i\right)\\ &= y_1 + (y_2 - a_0 y_1)z + (y_3 - a_0 y_2 - a_1 y_1)z^2\\ &\quad + (y_4 - a_0 y_3 - a_1 y_2 - a_2 y_1)z^3 + \cdots\\ &= \sum_{j=0}^{\infty}\left(\sum_{i=j}^{\infty} a_i y_{j-i}\right) z^j = k(z),\end{aligned}$$

where we have changed the order of addition appropriately, which is allowed since all the series above converge absolutely. Thus, as long as $1 - zh(z) \ne 0$, we have

$$\sum_{i=0}^{\infty} y_{i+1} z^i = \frac{k(z)}{1-zh(z)}. \tag{4.1.1}$$

On the other hand, we have $q^{-1}h(q^{-1}) = 1$ and that q^{-1} is the solution for $zh(z) = 1$ which has the smallest modulus by Lemma 1.2. Hence, for $|z| < |q|^{-1}$, we have (4.1.1). This completes the proof of Theorem 4.1. □

Now Theorem 1.5 follows from Theorems 1.4 and 4.1.

Proof of Theorem 1.7: By Theorem 1.3 and Lemma 2.1, if $|q| \ge 1$, then the series $\sum_{n=1}^{\infty} |y_n|$ does not converge. Suppose that $|q| < 1$. The radius of convergence of the power series

$$c(z) = \sum_{i=0}^{\infty} |y_{i+1}| z^i$$

is equal to the radius of convergence R'' of the power series

$$\sum_{i=0}^{\infty} y_{i+1} z^i .$$

By Theorem 4.1 together with our assumption, we have $R'' \ge |q|^{-1} > 1$. Thus, the series $c(z)$ for $z = 1$ converges. Then the rest of Theorem 1.7 follows from Theorem 4.1. This completes the proof. □

Corollary 4.2: Let $\{g_i\}_{i=1}^{\infty}$ be the ∞-generalized Fibonacci sequence associated with the initial sequence $\{g_{-i}\}_{i=0}^{\infty}$, where $g_0 = 1$ and $g_{-i} = 0$ for $i \ge 1$. If $|q| < 1$, then the series $\sum_{i=1}^{\infty} g_i$ converges to

$$\sum_{i=0}^{\infty} a_i \Bigg/ \left(1 - \sum_{i=0}^{\infty} a_i\right).$$

5. CONCLUDING REMARKS

In this section we give some remarks about questions (Q1)-(Q5) raised in §1.

About (Q1), in the finite case, the answer to this question is given by a Binet-type formula (e.g., see [1]). The question in the infinite case is also posed in [4, Problem 4.5]. In a forthcoming paper we will consider approximation of ∞-generalized Fibonacci sequences by finitely generalized ones and will give an asymptotic Binet formula which will give an answer to the question in a certain sense. This study is also closely related to question (Q2).

About (Q2), in the finite case, it has been shown that if the characteristic polynomial has a simple root of maximal modulus then the sequence is asymptotically geometric (see [1]). This condition is satisfied as long as the leading weight coefficient a_0 has sufficiently large modulus (see Theorem 15 and Remark 16 of [1]). In [4], the authors have shown that a statement similar to this also holds in the infinite case as well, which is nothing but Theorem 1.3 of the present paper.

About (Q3) and (Q4), Theorems 1.4 and 1.5, respectively, give satisfactory answers in the nonnegative real coefficient case under our assumption. In the general case, Theorems 1.6 and 1.7, respectively, give partial answers to the questions.

About (Q5), in a forthcoming paper, combinatorial expressions for the general terms of an ∞-generalized Fibonacci sequence will be studied.

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