∞-GENERALIZED FIBONACCI SEQUENCES AND MARKOV CHAINS

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1. INTRODUCTION

Let $\{a_j\}_{j=0}^{r-1}$ $(r \geq 2, a_{r-1} \neq 0)$ be a sequence of real numbers. An r-generalized Fibonacci sequence $\{V_n\}_{n=0}^{+\infty}$ is defined by the following linear recurrence relation of order r:

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1} \quad \text{for } n \geq r-1,$$

where $V_0, \ldots, V_{r-1}$ are specified by the initial conditions. Such sequences are largely studied in the literature (see, e.g., [2], [3], [6], and [7]). Let $\{a_j\}_{j \geq 0}$ be a sequence of real numbers and consider the sequence $\{V_n\}_{n \in Z}$ defined by the following linear recurrence relation of order ∞:

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_m V_{n-m} + \cdots, \text{ for } n \geq 0, \tag{1}$$

where $\{V_{-j}\}_{j \geq 0}$ are specified by the initial conditions. Such sequences, called *∞-generalized Fibonacci sequences*, were introduced and studied in [8]. We shall refer to them in the sequel as *sequences (1)*.

The aim of this paper, motivated by [8] and [10], is to study the connection between sequences (1) and Markov chains when the coefficients $\{a_j\}_{j \geq 0}$ are nonnegative. Such a connection is a generalization of those considered in [9] for r-generalized Fibonacci sequences. As in [8], we consider some hypotheses on $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$ in order to ensure the existence of the general term V_n for any $n \geq 1$, and then we extend results of [3] and [9] to the case of sequences (1). More precisely, using some Markov chain properties (see [1], [4], and [5]), we give a necessary and sufficient condition on the convergence of the ratio $\frac{V_n}{q^n}$, where $q > 0$ is a specified real number. This result extends the sufficient conditions of [8], under the hypotheses considered on the two sequences $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$. We also give the expression $\lim_{n \to +\infty} \frac{V_n}{q^n}$.

This paper is organized as follows. In Section 2 we study the case of sequences (1) in connection with Markov chains, when the coefficients $\{a_j\}_{j \geq 0}$ are nonnegative with $\sum_{j \geq 0} a_j = 1$. We also give a necessary and sufficient condition for the convergence of V_n and the expression of $\lim_{n \to +\infty} V_n$. In Section 3 we extend the results of Section 2 to the case of arbitrary nonnegative coefficients.

2. SEQUENCES (1) AND MARKOV CHAINS

2.1 Fundamental Hypotheses and Existence of the General Term

Let $\{V_n\}_{n \in Z}$ be a sequence (1). Its general term V_n does not exist in general for any $n \geq 1$. For example, suppose that $\{a_j\}_{j \geq 0}$ and $\{V_{-j}\}_{j \geq 0}$ are defined by

$$a_0 = 1,\ a_j = j^j \text{ for } j \geq 1 \text{ and } V_0 = 1,\ V_{-j} = j^{-(j+2)} \text{ for } j \geq 1.$$

Then, by a direct computation, we obtain

$$V_1 = 1 + \sum_{j\geq 1} \frac{1}{j^2} \quad \text{and} \quad V_2 = V_1 + V_0 + \sum_{m\geq 2} \frac{m^m}{(m-1)^{m+1}} = +\infty.$$

Thus, to ensure the existence of V_n for any $n \geq 0$, we need some hypotheses on the two sequences $\{a_j\}_{j\geq 0}$ and $\{V_{-j}\}_{j\geq 0}$. More precisely, suppose that the following hypotheses are satisfied:

(H.1) For any $m \geq 0$, there exists $k \geq m$ such that $a_k > 0$.

(H.2) There exists $C > 0$ such that $a_k \leq C$.

(H.3) The series $\sum_{m\geq 0} |V_{-m}|$ is convergent.

The two hypotheses (H.2) and (H.3) are trivially satisfied in the case of r-generalized Fibonacci sequences. These three hypotheses are more convenient with a Markov chain formulation of sequences (1). They are not necessary for the existence of the general term V_n. Other conditions are considered in [8].

2.2 Sequences (1) Whose Coefficients Are Nonnegative with Sum 1

Suppose that the coefficients $\{a_j\}_{j\geq 0}$ of the sequence (1) satisfy hypothesis (H.1) and the following condition:

$$\sum_{j\geq 0} a_j = 1. \tag{2}$$

It is obvious that identity (2) implies (H.2) is trivially satisfied. Consider the following matrix:

$$P = \begin{pmatrix} a_0 & a_1 & \cdots & a_n & \cdots & & & \\ 1 & 0 & 0 & \cdots & 0 & \cdots & & \\ 0 & 1 & 0 & 0 & \cdots & 0 & \cdots & \\ \vdots & & & & & & & \\ 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & \\ \vdots & & & & & & & \end{pmatrix}. \tag{3}$$

Condition (2) shows that the matrix P defined by (3) is a stochastic matrix. Then P is a transition matrix of a Markov chain $(\mathcal{T})$, whose state space is $\mathbf{N} = \{0, 1, \ldots\}$. Set $P = (P(n, m))_{n, m \in \mathbf{N}}$, then $P(0, m) = a_m$ and $P(n, m) = \delta_{n-1, m}$ for $n \geq 1$, where $\delta_{k,s}$ is the Kronecker symbol. Set $P^k = P \cdots P$ (k times), then $P^k = (P^{(k)}(n, m))_{n, m \in \mathbf{N}}$ for any $k \geq 1$, where $P^{(k)}(n, m)$ is the probability to go from the state n to the state m after k transitions. Since $P(n, n-1) = 1$, we derive

$$P^{(n-m)}(n, m) = 1 \text{ for any } m < n. \tag{4}$$

Then we have the following proposition.

Proposition 2.1: Let $\{a_j\}_{j\geq 0}$ be a sequence of nonnegative real numbers such that hypothesis (H.1) and condition (2) are satisfied. Let $(\mathcal{T})$ be the Markov chain associated to the matrix P defined by (3). Then:

(i) The chain $(\mathcal{T})$ is irreducible.

(ii) The chain $(\mathcal{T})$ is recurrent positive if $\sum_{m\geq 0}(m+1)a_m < +\infty$ and it is recurrent null if $\sum_{m\geq 0}(m+1)a_m = +\infty$.

Proof: (i) Let n and m be two states of $(\mathcal{T})$. Suppose, for example, that $m < n$. Hypothesis (H.1) and relation (4) imply that there exists $n_0 > n$ such that $a_{n_0} > 0$ and thus

$$P^{(m+n_0+1-n)}(n,m) \geq P^{(m)}(m,0)P(0,n_0)P^{(n_0-n)}(n_0,n)$$

which implies that $P^{(m+n_0+1-n)}(n,m) \geq a_{n_0} > 0$. Hence, the Markov chain $(\mathcal{T})$ is irreducible.

(ii) To study the nature of $(\mathcal{T})$, it is sufficient to study the nature of the state 0. Starting from 0, the Markov process associated to $(\mathcal{T})$ will go at the first transition to a state m with probability a_m. And it will be back to 0 with probability 1 after m transitions. Therefore, a_m is the probability of going from 0 and coming back to this state after $m+1$ transitions. The probability of coming back to 0 is $\sum_{m=0}^{+\infty} a_m = 1$. Therefore, $(\mathcal{T})$ is recurrent. Let T_0 be the real random variable which defines the first instant of return of the process to 0. We have established that $a_m = \Pr\{T_0 = m+1\}$; thus, the mean value of T_0 is $E(T_0) = \sum_{m=0}^{+\infty}(m+1)a_m$. Then $(\mathcal{T})$ is recurrent positive if $\sum_{m\geq 0}(m+1)a_m < +\infty$ and it is recurrent null if $\sum_{m\geq 0}(m+1)a_m = +\infty$. □

Remark 2.1: Let R be the radius of convergence of the series $\sum_{m\geq 0} a_m X^m$. Hypothesis (H.2) implies that $R \geq 1$. R is also the radius of convergence of the series $\sum_{m\geq 0} m a_m X^m$. Hence, if $R > 1$, we have $\sum_{m\geq 0} m a_m < +\infty$. Then $(\mathcal{T})$ is recurrent positive.

Recall that the period $d(m)$ of a given state m of $(\mathcal{T})$ is defined by

$$d(m) = \mathrm{CGD}\{k \in \mathbf{N};\ P^{(k)}(n,m) > 0\}.$$

It is well known that, for an irreducible Markov chain $(\mathcal{T})$, we have $d(m) = d(0) = d$ for any m in $(\mathcal{T})$ (see, e.g., [4]). We recall here a very well-known theorem on the asymptotic behavior of a Markov chain.

Theorem 2.2: (See, e.g., [4].) Let $P = (P(n,m))_{n,m\in\mathbf{N}}$ be the transition matrix of an irreducible Markov chain $(\mathcal{T})$. Then:

(i) The sequence of matrices $\{P^k\}_{k\geq 0}$ converges if and only if the Markov chain $(\mathcal{T})$ is aperiodic or identically $d = 1$.

(ii) If $(\mathcal{T})$ is recurrent null, then $\lim_{k\to+\infty} P^{(k)}(n,m) = 0$ for any states n and m in $(\mathcal{T})$.

(iii) If $(\mathcal{T})$ is recurrent positive, then $\lim_{k\to+\infty} P^{(k)}(n,m)$ does not depend on n and we have $\lim_{k\to+\infty} P^{(k)}(n,m) = \Pi(m)$, where $\Pi(m) > 0$ for any m. And the stationary distribution vector $\Pi = (\Pi(0), \Pi(1), \ldots, \Pi(m), \ldots)$ is the solution of the following matrix equation

$$\Pi = \Pi \cdot P, \tag{5}$$

where $\sum_{m=0}^{+\infty} \Pi(m) = 1$.

Let $\{V_n\}_{n\in\mathbf{Z}}$ be a sequence (1) and consider the infinite column vector $X_n = (V_n, V_{n-1}, \ldots, V_{n-k}, \ldots)^t$, where R^t means the transpose of R. We can show easily that expression (1) may be written as follows:

$$X_{n+1} = PX_n \quad \text{or} \quad X_{n+1} = P^{n+1}X_0 \tag{6}$$

for any $n \geq 0$, where $X_0 = (V_0, V_{-1}, \ldots, V_{-k}, \ldots)^t$ is the infinite vector of the initial conditions. With the use of (6), Proposition 2.1, and Theorem 2.2, we can extend the necessary and sufficient condition of convergence established in [3] and [9] for r-generalized Fibonacci sequences to the case of sequences (1) as follows.

Theorem 2.3: Let $\{a_j\}_{j\geq 0}$ and $\{V_{-j}\}_{j\geq 0}$ be two sequences of real numbers such that hypotheses (H.1) and (H.3) and condition (2) are satisfied. Then the associated sequence (1) converges if and only if the following condition $(\mathscr{C})$: $\mathrm{CGD}\{j+1; a_j>0\}=1$ is satisfied, where CGD means the common great divisor.

Proof: From (6), we derive that the sequence (1) converges for any choice of the initial conditions $\{V_{-j}\}_{j\geq 0}$ if and only if the sequence of matrices $\{P^k\}_{k\geq 0}$ converges. Then, let us study the aperiodicity of the Markov chain $(\mathscr{T})$ associated with the matrix P. We search the period of the state 0. Starting from state 0, the process will go at the first transition to a state m with probability $a_m>0$. And it comes back to 0 after m transitions with probability 1. Hence, the process returns to 0 after $m+1$ transitions. Starting from the state 0 after crossing k_1 times the state m_1, k_2 times the state m_2, ... Then the process has made $k_1(m_1+1)+k_2(m_2+1)+\cdots$ transitions. Thus, $P^{(n)}(0,0)>0$ if and only if n is of the following form: $n=k_1(m_1+1)+k_2(m_2+1)+\cdots$, where k_1, $k_2,\ldots$ are in N. Hence, we have $\{n; P^{(n)}(0,0)>0\}=\{n; n=k_1(m_1+1)+k_2(m_2+1)+\cdots;\ k_1,\ k_2,\ldots\in\mathrm{N}\}$, which implies that $\mathrm{CGD}\{n;\ P^{(n)}(0,0)>0\}=\mathrm{CGD}\{m+1;\ a_m>0\}$. Then $(\mathscr{T})$ is aperiodic if and only if condition $(\mathscr{C})$ is satisfied. Thus, the sequence (1) converges if and only if condition $(\mathscr{C})$ is satisfied. □

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4: Under the hypotheses of Theorem 2.3 and if $a_0>0$, then the sequence (1) converges.

Now we shall find the expression of the limit of the sequence (1) when condition $(\mathscr{C})$ is satisfied.

Lemma 2.5: Let $\{a_j\}_{j\geq 0}$ be a sequence of real numbers such that hypothesis (H.1) and the two conditions (2) and $(\mathscr{C})$ are verified. Let $P=(P(n,m))_{n,m\geq 0}$ be the stochastic matrix (3). Then we have:

(i) $\lim_{n\to+\infty}P^{(k)}(n,m)=0$ if $\sum_{j=0}^{+\infty}(j+1)a_j=+\infty$.

(ii) $\lim_{n\to+\infty}P^{(k)}(n,m)=\Pi(m)$ if $\sum_{j=0}^{+\infty}(j+1)a_j<+\infty$, where

$$\Pi(m)=\frac{\sum_{l=m}^{+\infty}a_l}{\sum_{l=0}^{+\infty}(l+1)a_l}. \tag{7}$$

Proof: Proposition 2.1 shows that $(\mathscr{T})$ is irreducible. And condition $(\mathscr{C})$ implies that $(\mathscr{T})$ is aperiodic and $\lim_{n\to+\infty}P^{(k)}(n,m)$ exists. Proposition 2.1, Theorem 2.2, and condition $(\mathscr{C})$ allow us to see that (i) $(\mathscr{T})$ is recurrent null with $\lim_{n\to+\infty}P^{(k)}(n,m)=0$ if $\sum_{m=0}^{+\infty}(m+1)a_m=+\infty$, and (ii) $(\mathscr{T})$ is recurrent positive with $\lim_{n\to+\infty}P^{(k)}(n,m)=\Pi(m)$ if $\sum_{m=0}^{+\infty}(j+1)a_j<+\infty$, where $\Pi(m)$ is the $(m+1)^{\text{th}}$ component of the stationary distribution vector $\Pi=(\Pi(0),\Pi(1),\ldots,\Pi(k),\ldots)$ which satisfies

$$\Pi=\Pi\cdot P \quad\text{and}\quad \sum_{m=0}^{+\infty}\Pi(m)=1. \tag{8}$$

The first equation of (8) is equivalent to an infinite system of equations whose unknown variables are $\Pi(m)$. By taking into consideration the second equation of (8), we derive

$$\Pi(m) = \frac{\sum_{l=m}^{+\infty} a_l}{\sum_{l=0}^{+\infty}(l+1)a_l}. \quad \square$$

Theorem 2.6: Under the hypotheses of Theorem 2.3 and if condition ($\mathscr{C}$) is satisfied, we have:

(i) $\lim_{n\to+\infty} V_n = 0$ if $\sum_{j=0}^{+\infty}(j+1)a_j = +\infty$.

(ii) $\lim_{n\to+\infty} V_n = \sum_{j=0}^{+\infty}\Pi(m)V_{-m}$ if $\sum_{j=0}^{+\infty}(j+1)a_j < +\infty$, where the $\Pi(m)$ are given by (7).

Proof: Expression (6) shows that $V_k = \sum_{m=0}^{+\infty} P^{(k)}(0,m)V_{-m}$. The inequality $|P^{(k)}(0,m)V_{-m}| \leq |V_{-m}|$ and hypothesis (H.3) imply that

$$\lim_{k\to+\infty} V_k = \sum_{m=0}^{+\infty}(\lim_{k\to+\infty} P^{(k)}(0,m))V_{-m}.$$

Hence, using Lemma 2.5, we derive the result. $\square$

Theorem 2.6 is a generalization of Theorem 2.4 of [9] to the case of sequences (1) under hypotheses (H.1), (H.2), and (H.3).

2.3 The Case of CGD$\{m+1; a_m > 0\} \geq 2$

Let $\{a_j\}_{j\geq 0}$ be a sequence of nonnegative real numbers which satisfies hypothesis (H.1) and condition (2). Suppose that CGD$\{m+1; a_m > 0\} \geq 2$. Let P be the stochastic matrix (3), and consider the Markov chain ($\mathscr{T}$) associated with P. Then we have the following proposition.

Proposition 2.7: (See, e.g., [5].) Let ($\mathscr{T}$) be an irreducible recurrent positive Markov chain. Let d be the period of the states of ($\mathscr{T}$). Suppose that $d \geq 2$. Then the state space E of ($\mathscr{T}$) may be written as follows: $E = D_1 \cup D_2 \cup \cdots \cup D_d$, where $D_i \cap D_j = \emptyset$ for $i \neq j$, such that if the process is in the class D_i at the instant n, then it can go to the class D_{i+1} after one transition, with probability 1 (for $i = d$, it goes from D_d to D_1). Each class D_i $(1 \leq i \leq d)$ is called a cyclic class. For any k, l with $k \leq l \leq d$ and i, j in E, the following limit, $\lim_{n\to+\infty} P^{(nd+l)}(i,j)$, exists and for any $i \in D_k$ we have

$$\lim_{n\to+\infty} P^{(nd+l)}(i,j) = \begin{cases} d\Pi(j), & \text{if } j \in D_{k+l} \ (\text{mod}\, d), \\ 0, & \text{if not,} \end{cases}$$

where $\Pi(j)$ is the $(j+1)^{\text{th}}$ component of the stationary distribution vector of P.

In our case, we have $P(i+1,i) = 1$; hence, the cyclic classes are given by $D_j = \{nd+j; n \geq 0\}$, $j = 0, 1, \ldots, d$. We derive the following result from Proposition 2.7.

Theorem 2.8: Under the hypotheses of Theorem 2.3, suppose that the Markov chain ($\mathscr{T}$) associated with P is irreducible recurrent positive. Suppose that CGD$\{m+1; a_m > 0\} \geq 2$. Then $\{V_n\}_{n\in\mathbf{Z}}$, the sequence (1) has d subsequences defined by $\{V_{nd+l}\}_{n\in\mathbf{Z}}$, where $l = 0, 1, \ldots, d-1$, and each of these subsequences, which is also a sequence (1), is convergent. More precisely, for any arbitrary initial conditions and a fixed l $(0 \leq l \leq d-1)$, we have

$$\lim_{n\to+\infty} V_{nd+l} = d\sum_{k=0}^{+\infty}\Pi(kd+l)V_{-(kd+l)},$$

where the $\Pi(m)$ are given by expression (7).

Proof: We have $V_{nd+l} = \sum_{m=0}^{+\infty} P^{(nd+l)}(0, m)V_{-m}$. Hypothesis (H.3) implies that

$$\lim_{n\to+\infty} V_{nd+l} = \sum_{m=0}^{+\infty} (\lim_{n\to+\infty} P^{(nd+l)}(0, m))V_{-m},$$

and the result is derived from Proposition 2.7. □

Theorem 2.8 is an extension of Theorem 4.2 of [9] to sequences (1), whose nonnegative coefficients satisfy condition (2), under hypotheses (H.1) and (H.3).

3. SEQUENCES (1) WHOSE COEFFICIENTS ARE NONNEGATIVE

In this section we consider that the nonnegative coefficients $\{a_j\}_{j\geq 0}$ are of arbitrary finite sum.

Let $\{V_n\}_{n\in\mathbf{Z}}$ be a sequence (1) such that hypotheses (H.1), (H.2), and (H.3) are satisfied. Let R be the radius of convergence of the power series

$$f(x) = \sum_{m=0}^{+\infty} a_m x^{m+1}. \tag{9}$$

Hypothesis (H.2) implies that $R \geq 1$.

Consider the following limit $L = \lim_{x\to R^-} f(x)$. The study of sequences (1) depends on the following three cases: $L < 1$, $L = 1$, and $L > 1$.

Study of the Case $L < 1$. In this case, we have $\sum_{m=0}^{+\infty} a_m < 1$ because $R \geq 1$ and the function f is not decreasing. Then we have the following result.

Proposition 3.1: Let $\{V_n\}_{n\in\mathbf{Z}}$ be a sequence (1) such that hypotheses (H.1), (H.2), and (H.3) are satisfied. Then, if $\sum_{m=0}^{+\infty} a_m < 1$, we have $\lim_{n\to+\infty} V_n = 0$ for any choice of the initial conditions.

Proof: Let $S_N = \sum_{n=1}^{N} |V_n|$. We have $S_N = \sum_{n=1}^{N} |\sum_{m=0}^{+\infty} a_m V_{n-m-1}| \leq \sum_{n=1}^{N} \sum_{m=0}^{+\infty} a_m |V_{n-m-1}|$, which implies that $S_N \leq \sum_{m=0}^{+\infty} a_m (\sum_{k=0}^{+\infty} |V_{-k}| + S_N)$. Thus, $S_N \leq (\sum_{m=0}^{+\infty} a_m)(\sum_{k=0}^{+\infty} |V_{-k}|)(1 - \sum_{m=0}^{+\infty} a_m)^{-1}$. And from hypothesis (H.3), we derive $\lim_{n\to+\infty} V_n = 0$. □

Study of the Case $L = 1$. In this case, we have the following two subcases: $\sum_{m=0}^{+\infty} a_m = 1$ if $R = 1$ and $\sum_{m=0}^{+\infty} a_m < 1$ if $R > 1$. The first one is studied in Section 2 and the second one is nothing but the preceding case.

Study of the Case $L > 1$. In this case, the analytic power series of f defined by (9) is a continuous and not decreasing function on $]0, R[$ that satisfies $f(0) = 0$. Then there exists $x_0 \in\ ]0, R[$ such that $f(x_0) = 1$. Set $q = 1/x_0$ and $b_m = a_m / q^{m+1}$ for any $m \in \mathbf{N}$. Then we have

$$b_m \geq 0,\ \sum_{m=0}^{+\infty} b_m = 1, \text{ and } \mathrm{CGD}\{m+1;\ a_m > 0\} = \mathrm{CGD}\{m+1;\ b_m > 0\}. \tag{10}$$

Now consider the sequence $\{W_n\}_{n\in\mathbf{Z}}$ defined by $W_n = \frac{V_n}{q^n}$. From relation (1), we derive

$$W_{n+1} = \sum_{m=0}^{+\infty} b_m V_{n-m} \tag{11}$$

for any $n \geq 0$. Thus, $\{W_n\}_{n \in Z}$ is also a sequence (1) that satisfies the two hypotheses (H.1) and (H.2) and condition (2). Hypothesis (H.3) is not satisfied in general by the initial conditions $\{W_{-m}\}_{m \geq 0}$. From Theorems 2.3 and 2.6 and expressions (10) and (11), we can formulate the extension of Theorems 5 and 9 of [3] and Theorems 3.1 and 3.3 of [9] as follows.

Theorem 3.2: Let $\{V_n\}_{n \in Z}$ be a sequence (1) such that hypotheses (H.1), (H.2), and (H.3) are satisfied. Let $\{W_n\}_{n \in Z}$ be the sequence defined by (11) and suppose that the initial conditions $\{W_{-m}\}_{m \geq 0}$ satisfy hypothesis (H.3). Then:

(a) $\lim_{n \to +\infty} W_n = \lim_{n \to +\infty} \frac{V_n}{q^n}$ exists if and only if condition ($\mathscr{C}$) is satisfied.

(b) If condition ($\mathscr{C}$) is satisfied, we have:

(i) $\lim_{n \to +\infty} \frac{V_n}{q^n} = 0$ if $\sum_{m=0}^{+\infty}(m+1)b_m = +\infty$;

(ii) $\lim_{n \to +\infty} \frac{V_n}{q^n} = \sum_{m=0}^{+\infty} \Pi(m) V_{-m} q^m$ if $\sum_{m=0}^{+\infty}(m+1)b_m < +\infty$, where

$$\Pi(m) = \frac{\sum_{s=m}^{+\infty} b_s}{\sum_{s=0}^{+\infty}(s+1)b_s}. \tag{12}$$

The second expression of $\lim_{n \to +\infty} \frac{V_n}{q^n}$ given in Theorem 3.2 is identical to the expression of Theorem 3.10 in [8].

For $q \leq 1$ or $\sum_{s=0}^{+\infty} \leq 1$, we have $|W_m| = |V_m q^m| \leq |V_m|$. Thus, hypothesis (H.3) is satisfied by $\{W_{-m}\}_{m \geq 0}$. But for $q > 1$ or $\sum_{s=0}^{+\infty} a_s > 1$, such hypothesis is not satisfied in general by $\{W_{-m}\}_{m \geq 0}$.

Case $d = \text{CGD}\{m+1;\ a_m > 0\} \geq 2$. In this case, we derive from expression (10) that $\text{CGD}\{m+1;\ b_m > 0\} \geq 2$. Thus, we can extend Theorem 4.2 of [9] as follows.

Theorem 3.3: Under the hypotheses of Theorem 3.2, suppose that $d = \text{CGD}\{m+1;\ a_m > 0\} \geq 2$. Then $\{W_n\}_{n \in Z}$ has d subsequences defined by $\{W_{nd+l}\}_{n \in Z}$, $l = 0, 1, \dots, d-1$ that are also sequences (1). And each subsequence $\{W_{nd+l}\}_{n \in Z}$ converges for any choice of those initial conditions with

$$\lim_{n \to +\infty} W_{nd+l} = \lim_{n \to +\infty} \frac{V_{nd+l}}{q^{nd+l}} = d \sum_{k=0}^{+\infty} \Pi(kd+l) W_{-(kd+l)},$$

where the $\Pi(kd+l)$ are given by expression (12).

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