ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-567 ***Proposed by Ernst Herrmann, Siegburg, Germany***

Let F_n denote the n^{th} Fibonacci number. For any natural number $n \geq 3$, the four inequalities

$$\begin{aligned}\frac{1}{F_n}+\frac{1}{F_{n+a_1}}&<\frac{1}{F_{n-1}}\\ &\leq\frac{1}{F_n}+\frac{1}{F_{n+a_1-1}},\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{1}{F_n}+\frac{1}{F_{n+a_1}}+\frac{1}{F_{n+a_1+a_2}}&<\frac{1}{F_{n-1}}\\ &\leq\frac{1}{F_n}+\frac{1}{F_{n+a_1}}+\frac{1}{F_{n+a_1+a_2-1}},\end{aligned} \tag{2}$$

determine uniquely two natural numbers a_1 and a_2.

Find the numbers a_1 and a_2 dependent on n.

H-568 ***Proposed by N. Gauthier, Royal Military College of Canada, Kingston, Ontario***

The following was inspired by Paul S. Bruckman's Problem B-871 in *The Fibonacci Quarterly* (proposed in Vol. 37, no. 1, February 1999; solved in Vol. 38, no. 1, February 2000).

"For integers $n, m \geq 1$, prove or disprove that

$$f_m(n) \equiv \frac{1}{\binom{2n}{n}^2}\sum_{k=0}^{2n}\binom{2n}{n}^2 |n-k|^{2m-1}$$

is the ratio of two polynomials with integer coefficients

$$f_m(n) = P_m(n) / Q_m(n),$$

where $P_m(n)$ is of degree $\lfloor \frac{3m}{2} \rfloor$ in n and $Q_m(n)$ is of degree $\lfloor \frac{m}{2} \rfloor$; determine $P_m(n)$ and $Q_m(n)$ for $1 \leq m \leq 5$."

H-569 ***Proposed by Paul S. Bruckman, Berkeley, CA***

Let $\tau(n)$ and $\sigma(n)$ denote, respectively, the number of divisors of the positive integer n and the sum of such divisors. Let $e_2(n)$ denote the highest exponent of 2 dividing n. Let p be any odd prime, and suppose $e_2(p+1)=h$. Prove the following for all odd positive integers a:

$$e_2(\sigma(p^a)) = e_2(\tau(p^a)) + h - 1. \qquad (*)$$

SOLUTIONS

Bi-Nomial

H-555 ***Proposed by Paul S. Bruckman, Berkeley, CA***
(Vol. 37, no. 3, August 1999)

Prove the following identity:

$$\begin{aligned}&(x^n+y^n)(x+y)^n \\ &= -(-xy)^n + \sum_{k=0}^{[n/3]} (-1)^k C_{n,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n-3k}, \qquad (1)\\ &n = 1, 2, \ldots,\end{aligned}$$

where

$$C_{n,k} = \binom{n-2k}{k} \cdot n/(n-2k).$$

Using (1), prove the following:

(a) $5^{n/2}L_n = -1 + \sum_{k=0}^{[n/3]} (-1)^k C_{n,k} 5^k 4^{n-3k}, \quad n = 2, 4, 6, \ldots;$

(b) $5^{(n+1)/2}F_n = 1 + \sum_{k=0}^{[n/3]} (-1)^k C_{n,k} 5^k 4^{n-3k}, \quad n = 1, 3, 5, \ldots;$

(c) $L_n = -1 + \sum_{k=0}^{[n/3]} (-1)^k C_{n,k} 2^{n-3k}, \quad n = 1, 2, 3, \ldots.$

Solution by Reiner Martin, New York, NY

Let us write

$$P_n(x,y) = (x^n+y^n)(x+y)^n + (-xy)^n.$$

We have

$$P_{n+3}(x,y) = P_{n+2}(x,y)(x^2+xy+y^2) - P_n(x,y)[xy(x+y)]^2.$$

Our goal is to show that the corresponding recursion holds for the sum in (1).

Next, note that

$$C_{n+3,k} = C_{n+2,k} + C_{n,k-1}.$$

Using this identity, we get

$$\sum_{k=0}^{[(n+3)/3]}(-1)^k C_{n+3,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n+3-3k}$$
$$=(x^2+xy+y^2)\sum_{k=0}^{[(n+2)/3]}(-1)^k C_{n+2,k}[xy(x+y)]^{2k}(x^2+xy+y^2)^{n+2-3k}$$
$$+[xy(x+y)]^{2k}\sum_{k=1}^{[n/3]+1}(-1)^k C_{n,k-1}[xy(x+y)]^{2(k-1)}(x^2+xy+y^2)^{n-3(k-1)}.$$

So, the sum in (1) satisfies the same recursion as $P_n(x,y)$. Since the cases $n=1,2,3$ are trivial, identity (1) follows for all $n\geq 1$.

Finally, (a) and (b) follow from (1) by specializing to $x=\alpha$ and $y=-\beta$, while (c) follows by using $x=\alpha$ and $y=\beta$.

Also solved by H.-J. Seiffert and the proposer.

Some Operator

H-556 *Proposed by N. Gauthier, Dept. of Physics, Royal Military College of Canada (Vol. 37, No. 4, November 1999)*

Let $f(x)$ and $g(x)$ be continuous and differentiable in the immediate vicinity of $x=a(\neq 0)$ and assume that, for some positive integer k,

$$f^{(n)}(a)=g^{(n)}(a)=0;\quad 0\leq n\leq k-1.$$

By definition,

$$f^{(n)}(x):=\frac{d^n}{dx^n}f(x)$$

for any continuous and differentiable function $f(x)$. Further, assume that one of the following conditions holds for $n=k$:

a. $f^{(k)}(a)\neq 0,\ g^{(k)}(a)=0;$

b. $f^{(k)}(a)=0,\ g^{(k)}(a)\neq 0;$

c. $f^{(k)}(a)\neq 0,\ g^{(k)}(a)\neq 0;$

Introduce the differential operator $D:=x\frac{d}{dx}$ and define, for m a nonnegative integer,

$$f_m(x):=D^m f(x),\quad g_m(x):=D^m g(x).$$

Prove that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f_k(a)}{g_k(a)},\quad a\neq 0.$$

Solution by the proposer

Note first that

$$D^2=x\frac{d}{dx}+x^2\frac{d^2}{dx^2};\quad D^3=x\frac{d}{dx}+3x^2\frac{d^2}{dx^2}+x^3\frac{d^3}{dx^3};\ \text{etc.},$$

the general term being, for $m\geq 1$,

$$D^m = \sum_{s=1}^{m} a_s(m) x^s \frac{d^s}{dx^s}, \tag{1}$$

as can readily be shown by induction on m. The set of coefficients $\{a_s(m): 1 \le s \le m; 1 \le m\}$ can be determined recursively, as follows. Consider

$$D^{m+1} = \sum_{s=1}^{m+1} a_s(m+1) x^s \frac{d^s}{dx^s}, \tag{2}$$

which follows from (1) with m replaced by $m+1$. But one also has that $D^{m+1} = D(D^m)$, where D^m is given by (1), so

$$\begin{aligned} D^{m+1} &= D\sum_{s=1}^{m} a_s(m) x^s \frac{d^s}{dx^s} \\ &= \sum_{s=1}^{m} a_s(m)\left[s x^s \frac{d^s}{dx^s} + x^{s+1} \frac{d^{s+1}}{dx^{s+1}} \right] \\ &= \sum_{s=1}^{m+1} [s a_s(m) + a_{s-1}(m)] x^s \frac{d^s}{dx^s}. \end{aligned} \tag{3}$$

The third line follows by introducing the definitions

$$a_s(m) = 0 : s > m; \quad a_0(m) = 0 : 1 \le m.$$

Equating the last line of (3) to (2) then gives the desired recurrence:

$$a_s(m+1) = s a_s(m) + a_{s-1}(m) : 1 \le s \le m+1, 1 \le m; \tag{4}$$

$a_0(m) = a_{m+1}(m) = 0$, $a_1(1) = 1$; $a_s(m)$ is thus a Stirling number of the second kind. Putting $s = m+1$ gives

$$a_{m+1}(m+1) = a_{m+1}(m) + a_m(m) = a_m(m), \tag{5}$$

so that $a_m(m) = 1$ by induction on m, since $a_1(1) = 1$. Now consider, for $k \ge 1$,

$$f_k(x) := D^k f(x) = \sum_{s=1}^{k} a_s(k) x^s \frac{d^s}{dx^s} f(x) = \sum_{s=1}^{k} a_s(k) x^s f^{(s)}(x)$$

and, similarly,

$$g_k(x) = \sum_{s=1}^{k} a_s(k) x^s g^{(s)}(x).$$

Evaluating at $x = a (\ne 0)$ and using $f^{(n)}(a) = g^{(n)}(a) = 0 : n = 0, 1, \dots, k-1$ for some k, with either one of conditions (a), (b), or (c) in the statement of the problem assumed to hold, then gives

$$f_k(a) = \sum_{s=1}^{k} a_s(k) a^s f^{(s)}(a) = a_k(k) a^k f^{(k)}(a) = a^k f^{(k)}(a); \ a \ne 0,$$

with an equivalent result for $g_k(a)$:

$$g_k(a) = a^k g^{(k)}(a); \ a \ne 0.$$

Finally, invoke l'Hôpital's rule to find the limit of f/g as $x \to a$ to get

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)} = \frac{f_k(a)}{g_k(a)}; \ a \ne 0.$$

This completes the required proof. Note, in passing, that this new formulation of l'Hôpital's rule makes it much easier to resolve indeterminate forms when $f(x)$ and $g(x)$ are polynomials. This is due to the fact that $D^k x^v = v^k x^v$ for v arbitrary and k a nonnegative integer.

Also solved by P. Bruckman and H.-J. Seiffert

Generalize

H-557 ***Proposed by Stanley Rabinowitz, Westford, MA (Vol. 37, no. 4, November 1999)***

Let $\langle w_n \rangle$ be any sequence satisfying the second-order linear recurrence $w_n = Pw_{n-1} - Qw_{n-2}$, and let $\langle v_n \rangle$ denote the specific sequence satisfying the same recurrence but with the initial conditions $v_0 = 2$, $v_1 = P$.

If k is an integer larger than 1, and $m = \lfloor k/2 \rfloor$, prove that, for all integers n,

$$v_n \sum_{i=0}^{m-1} (-Q^n)^i w_{(k-1-2i)n} = w_{kn} - (-Q^n)^m \times \begin{cases} w_0, & \text{if } k \text{ is even,} \\ w_n, & \text{if } k \text{ is odd.} \end{cases}$$

Note: This generalizes problem H-453.

Solution by Paul S. Bruckman, Berkeley, CA

Let

$$F(x; k, n) = \sum_{i=0}^{m-1} (-Q^n)^i x^{(k-1-2i)m}, \text{ where } m = [k/2]. \tag{1}$$

Then, after simplification,

$$F(x; k, n) = x^{(k-m)n}\{x^{mn} - (-1)^m Q^{mn} x^{-mn}\} / \{x^n + Q^n x^{-n}\}. \tag{2}$$

Let $\langle u_n \rangle$ denote the *fundamental sequence* associated with the given recurrence, that is, the sequence satisfying this same recurrence but with initial conditions $u_0 = 0$, $u_1 = 1$. Then $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, $v_n = \alpha^n + \beta^n$, where $\alpha = (P+\theta)/2$, $\beta = (P-\theta)/2$, and $\theta = (P^2 - 4Q)^{1/2}$. Note that $\alpha + \beta = P$ and $\alpha\beta = Q$.

We readily determine the following results:

$$F(\alpha; k, n) = \alpha^{(k-m)n}(\alpha^{mn} - (-1)^m \beta^{mn}) / v_n; \tag{3}$$

$$F(\beta; k, n) = (-1)^{m+1} \beta^{(k-m)n}(\alpha^{mn} - (-1)^m \beta^{mn}) / v_n. \tag{4}$$

Next, we define the following sums:

$$G_w(k, n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i w_{(k-1-2i)n}, \tag{5}$$

$$G_v(k, n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i v(k-1-2i)n. \tag{6}$$

Note that $G_v(k, n) = v_n\{F(\alpha; k, n) + F(\beta; k, n)\} = \{\alpha^{(k-m)n} - (-1)^m \beta^{(k-m)n}\}\{\alpha^{mn} - (-1)^m \beta^{mn}\}$ or, after simplification,

$$G_v(k, n) = v_{kn} - (-1)^m Q^{mn} v_{(k-2m)n}. \tag{7}$$

Note that $k = 2m$ if k is even, while $k = 2m+1$ if k is odd. Thus, we see that (7) is a special case of the statement of the problem, with $\langle w_n \rangle = \langle v_n \rangle$.

We now use the following relation between the general sequence $\langle w_n \rangle$ and the particular sequence $\langle v_n \rangle$:

$$w_N = \{w_n u_N - Q^n w_0 u_{N-n}\} / u_n. \tag{8}$$

This may be verified by noting that $u_{-N} = -Q^{-N} u_N$. In particular, we obtain

$$w_{kn} = \{w_n u_{kn} - Q^n w_0 u_{(k-1)n}\} / u_n. \tag{9}$$

Substituting the expression from (8) into the sum in (5), we obtain

$$u_n G_w(k,n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i \{w_n u_{(k-1-2i)n} - Q^n w_0 u_{n(k-2-2i)n} \text{ or}$$

$$u_n G_w(k,n) = w_n G_u(k,n) - Q^n w_0 G_u(k-1,n), \tag{10}$$

where

$$G_u(k,n) = v_n \sum_{i=0}^{m-1} (-Q^n)^i u_{(k-1-2i)n}. \tag{11}$$

Note that

$$\begin{aligned} G_u(k,n) &= v_n\{F(\alpha; k,n) - F(\beta; k,n)\} / (\alpha - \beta) \\ &= \{\alpha^{(k-m)n} + (-1)^m \beta^{(k-m)n}\}\{\alpha^{mn} - (-1)^m \beta^{mn}\} / (\alpha - \beta) \\ &= \{\alpha^{kn} - \beta^{kn} - (-1)^m Q^{mn}(\alpha^{(k-2m)n} - \beta^{(k-2m)n})\} / (\alpha - \beta) \text{ or} \end{aligned}$$

$$G_u(k,n) = u_{kn} - (-1)^m Q^{mn} u_{(k-2m)n}. \tag{12}$$

We observe that (12) is another special case of the statement of the problem, with $\langle w_n \rangle = \langle u_n \rangle$.

Now, substituting the result of (12) into the expression in (10), we obtain the following:

$$\begin{aligned} u_n G_w(k,n) = {} & w_n u_{kn} - (-1)^m Q^{mn} w_n u_{(k-2m)n} \\ & - Q^n w_0 \{u_{(k-1)n} - (-1)^{m'} Q^{m'n} u_{(k-1-2m')n}\}, \end{aligned}$$

where $m' = [(k-1)/2]$.

(a) If $k = 2m$, then $m' = m-1$ and $k = 2m'+2$. Then

$$\begin{aligned} u_n G_w(k,n) &= w_n u_{kn} - Q^n w_0 u_{(k-1)n} - (-1)^m Q^{mn} w_0 u_n \\ &= u_n w_{kn} - (-1)^m Q^{mn} w_0 u_n, \end{aligned}$$

using the result in (9). Hence, $G_w(k,n) = w_{kn} - (-1)^m Q^{mn} w_0$ if k is even.

(b) If $k = 2m+1$, then $m' = m$ and $k = 2m'+1$. Then

$$\begin{aligned} u_n G_w(k,n) &= w_n u_{kn} - Q^n w_0 u_{(k-1)n} - (-1)^m Q^{mn} w_n u_n \\ &= u_n w_{kn} - (-1)^m Q^{mn} w_n u_n, \end{aligned}$$

using the result in (9). Hence, $G_w(k,n) = w_{kn} - (-1)^m Q^{mn} w_n$ if k is odd.

We may combine both formulas into one, as follows:

$$G_w(k,n) = w_{kn} - (-1)^m Q^{mn} w_{(k-2m)n}. \tag{13}$$

Also solved by H.-J. Seiffert and the proposer.

❖❖❖