# ELEMENTARY PROBLEMS AND SOLUTIONS

## *Edited by*
## Russ Euler and Jawad Sadek

*Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.*

*If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.*

*Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".*

### BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

### PROBLEMS PROPOSED IN THIS ISSUE

**B-906** *Proposed by N. Gauthier, Royal Military College of Canada*

Consider the following $n \times n$ determinants,

$$\Delta_1(n) := \begin{vmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix};$$

$$\Delta_2(n) := \begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix};$$

$n$ is taken to be a positive integer and $\Delta_1(0)=1$, $\Delta_2(0)=0$, by definition. Prove the following:

a. $\Delta_1(n)=F_{2n+1}$;

b. $\Delta_2(n)=F_{2n}$.

**B-907** ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

Prove that

$$F_1^{F_1}\cdot F_2^{F_2}\cdot F_3^{F_3}\cdot\cdots\cdot F_n^{F_n}\le e^{(F_n-1)(F_{n+1}-1)}.$$

**B-908** ***Proposed by Indulis Strazdins, Riga Tech. University, Latvia***

The Fibonacci polynomials, $F_n(x)$, are defined by

$$F_0(x)=0,\ F_1(x)=1,\text{ and } F_{n+2}(x)=xF_{n+1}(x)+F_n(x)\quad\text{for } n\ge 0.$$

Prove the identity

$$F_{n+1}^2(x)-4xF_n(x)F_{n-1}(x)=x^2F_{n-2}^2(x)+(x^2-1)F_{n-1}(x)(xF_n(x)-F_{n-3}(x)).$$

**B-909** ***Proposed by J. Cigler, Universität Wien, Austria***

Consider an arbitrary sequence of polynomials $p_k(x)$ of the form $p_k(x)=x^{a_k}(x-1)^{b_k}$, where $a_k$ and $b_k$ are integers satisfying $a_k+b_k=k$ and $a_k\ge b_k+1\ge 0$. Let $L_{n,k}$ be the uniquely determined numbers such that $x^n=\sum L_{n,k}p_k(x)$. Show that

$$F_n=\sum L_{n,k}F_{a_k-b_k},$$

where $F_n$ are the Fibonacci numbers.

If all $a_k-b_k\in\{1,2\}$, then we have $F_n=\sum L_{n,k}$. This generalizes Proposition 2.2 of the paper "Fibonacci and Lucas Numbers as Cumulative Connection Constants" in *The Fibonacci Quarterly* **38.2** (2000):157-64.

**B-910** ***Proposed by Richard André-Jeannin, Cosnes et Romain, France***

Solve the equation $p^n+1=\frac{k(k+1)}{2}$, where $p$ is a prime number and $k$ is a positive integer.

***Remark:*** The case $p=2$ is Problem B-875 (*The Fibonacci Quarterly*, May 1999; see February 2000 for the solution).

## SOLUTIONS

### A Fibonacci Average Which Is a Lucas Number

**B-889** ***Proposed by Mario DeNobili, Vaduz, Lichtenstein***
***(Vol. 38, no. 1, February 2000)***

Find 17 consecutive Fibonacci numbers whose average is a Lucas number.

***Solution by Richard André-Jeannin, Cosnes et Romain, France***

The identity $S_n=F_n+\cdots+F_{n+16}=F_{n+18}-F_{n+1}$ follows easily from Binet's formulas. Therefore, we have to find integers $n$ and $m$ such that $F_{n+18}-F_{n+1}=17L_m$. This relation implies that $F_{n+18}-F_{n+1}\equiv 0 \pmod{17}$. By induction, one can verify that $F_{n+18}\equiv -F_n \pmod{17}$; thus, we have $F_{n+18}-F_{n+1}\equiv -F_n-F_{n+1}=-F_{n+2}\equiv 0 \pmod{17}$, which implies that $n+2\equiv 0 \pmod 9$, since 9 is the rank of apparition of 17.

Let us define the sequence $T_k$ by

$$T_k = \frac{S_{9K-2}}{17} = \frac{F_{9k+16} - F_{9k-1}}{17}.$$

Then we have

$$T_{-1} = \frac{S_{-11}}{17} = \frac{F_7 - F_{-10}}{17} = \frac{F_7 + F_{10}}{17} = 4 = L_3.$$

We shall prove that this is the only solution. It is straightforward to see that sequences such that $F_{9k+r}$, $L_{9k+r}$, or $T_k$ satisfy the recurrence

$$X_k = 76X_{k-1} + X_{k-2}. \tag{1}$$

Assuming first that $k \geq 0$, we see that $L_8 < T_0 < L_9$ *and that* $L_{17} < T_1 < L_{18}$. By this and (1), it is clear that $L_{9k+8} < T_k < L_{9k+9}$ for every $k \geq 0$. Thus, $T_k$ is not a Lucas number for $k \geq 0$.

On the other hand, we have

$$T_{-k} = (-1)^{k+1}\left(\frac{F_{9k-16} + F_{9k+1}}{17}\right).$$

by the formula $F_{-k} = (-1)^{k+1}F_k$. From this, we have

$$|T_{-k}| = \frac{F_{9k-16} + F_{9k+1}}{17}$$

for $k \geq 1$, and one can verify that $L_{11} < |T_{-2}| < L_{12}$ *and that* $L_{20} < |T_{-3}| < L_{21}$. Using this and (1), it is clear that $L_{9k-7} < |T_{-k}| < L_{9k-6}$ for $k \geq 2$. This concludes the proof.

***Also solved by Brian D. Beasley, David M. Bloom, Paul S. Bruckman, L. A. G. Dresel, H.-J. Seiffert, Indulis Strazdins, and the proposer.***

## A Sum of Products Equals Zero

**B-890** ***Proposed by Stanley Rabinowitz, Westford, MA***
***(Vol. 38, no. 1, February 2000)***

If $F_{-a}F_bF_{a-b} + F_{-b}F_cF_{b-c} + F_{-c}F_aF_{c-a} = 0$, show that either $a = b$, $b = c$, or $c = a$.

***Solution by Paul S. Bruckman, Berkeley, CA***

Let $U(a,b,c)$ denote the expression given in the statement of the problem. We prove the following identity:

$$U(a,b,c) = F_{a-b}F_{b-c}F_{c-a}. \tag{1}$$

Note that $F_n = 0$ iff $n = 0$; hence (assuming (1) is true), $U(a,b,c) = 0$ iff $a = b$, $b = c$, or $c = a$. Therefore, it suffices to prove (1).

The following known identities are employed:

$$5F_mF_n = L_{m+n} - (-1)^n L_{m-n}; \tag{2}$$

$$F_mL_n = F_{m+n} + (-1)^n F_{m-n}. \tag{3}$$

These lead to the following (symmetric) identity:

$$5F_xF_yF_z = F_{x+y+z} - (-1)^x F_{-x+y+z} - (-1)^y F_{x-y+z} - (-1)^z F_{x+y-z}. \tag{4}$$

In particular, $5F_{-a}F_bF_{a-b} = -(-1)^a F_{2a} + (-1)^b F_{2b} - (-1)^{a+b} F_{2b-2a}$.

Combining similar terms and simplifying, we obtain several cancellations, along with the following result:

$$5U(a,b,c) = -(-1)^{a+b}F_{2b-2a} - (-1)^{b+c}F_{2c-2b} - (-1)^{a+c}F_{2a-2c}. \tag{5}$$

On the other hand, $5F_{a-b}F_{b-c} = L_{a-c} - (-1)^{b+c}L_{a-2b+c}$; hence,

$$\begin{aligned} 5F_{a-b}F_{b-c}F_{c-a} &= F_{c-a}(L_{a-c} - (-1)^{b+c}L_{a-2b+c}) \\ &= F_0 + (-1)^{a+c}F_{2c-2a} - (-1)^{b+c}\{F_{2c-2b} + (-1)^{a+c}F_{2b-2z}\} \\ &= -(-1)^{a+b}F_{2b-2a} - (-1)^{b+c}F_{2c-2b} - (-1)^{a+c}F_{2a-2c}, \end{aligned}$$

which is seen to be the same expression as in (5). This establishes (1), hence the desired result.

*L. A. G. Dresel gave a solution similar to the featured one. However, he had a different proof for identity (1) based on his paper "Transformations of Fibonacci-Lucas Identities" in Applications of Fibonacci Numbers **5**:169-84.*

***Also solved by Brian D. Beasley, L. A. G. Dresel, Hradec Králové, and the proposer.***

## A Lucas-Pell Congruence

**B-891** ***Proposed by Aloysius Dorp, Brooklyn, NY***
***(Vol. 38, no. 1, February 2000)***

Let $\langle P_n \rangle$ be the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Find integers $a$, $b$, and $m$ such that $L_n \equiv P_{an+b} \pmod{m}$ for all integers $n$.

***Solution by H.-J. Seiffert, Berlin, Germany***

Extend the recursion of the Pell numbers to $n \in \mathbf{Z}$, and define the Pell-Lucas numbers by $Q_0 = 2$, $Q_1 = 2$, and $Q_{n+2} = 2Q_{n+1} + Q_n$ for $n \in \mathbf{Z}$. For the integers $a$ and $b$, where $a$ is odd, let

$$m = \gcd(Q_a - 1, P_b - 2, P_{a+b} - 1).$$

We claim that

$$L_n \equiv P_{an+b} \pmod{m} \ \text{ for all } n \in \mathbf{Z}.$$

Since $a$ is odd, we have [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):7-20, equation (3.29)]

$$P_{a(n+2)+b} = Q_a P_{a(n+1)+b} + P_{an+b}, \ \ n \in \mathbf{Z},$$

which by $Q_a \equiv 1 \pmod{m}$ implies that

$$P_{a(n+2)+b} \equiv P_{a(n+1)+b} + P_{an+b} \pmod{m}, \ \ n \in \mathbf{Z}.$$

Hence, if $A_n = L_n - P_{an+b}$, $n \in \mathbf{Z}$, then $A_{n+2} \equiv A_{n+1} + A_n \pmod{m}$, $n \in \mathbf{Z}$. Since $A_0 = 2 - P_b \equiv 0 \pmod{m}$ and $A_1 = 1 - P_{a+b} \equiv 0 \pmod{m}$, it now easily follows that $A_n \equiv 0 \pmod{m}$ for all $n \in \mathbf{Z}$. This proves the above stated congruence.

***Examples:***

***(a)*** With $a = -3$ and $b = 2$, we have $m = \gcd(-15, 0, 0) = 15$. The above result gives $L_n \equiv P_{-3n+2}$ for all $n \in \mathbf{Z}$.

***(b)*** Taking $a = 5$ and $b = 2$, we have $m = \gcd(81, 0, 168) = 3$. Hence, $L_n \equiv P_{5n+2} \pmod{3}$ for all $n \in \mathbf{Z}$.

*(c)* With $a = 5$ and $b = -6$, we have $m = \gcd(81, -72, 0) = 9$, so that $L_n \equiv P_{5n-6} \pmod 9$ for all $n \in \mathbf{Z}$.
*(d)* Take $a = 7$ and $b = -6$. Then, $m = \gcd(477, -72, 0) = 9$. Hence, $L_n \equiv P_{7n-6} \pmod 9$ for all $n \in \mathbf{Z}$.
*(e)* With $a = 9$ and $b = -8$, we have $m = \gcd(2785, -410, 0) = 5$, so that $L_n \equiv P_{9n-8} \pmod 5$ for all $n \in \mathbf{Z}$.

*The featured solution contains the solutions given by the other solvers.*

***Also solved by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, L. A. G. Dresel, and the proposer.***

## A Perfect Square Only When Modulo 47

**B-892** ***Proposed by Stanley Rabinowitz, Westford, MA***
***(Vol. 38, no. 1, February 2000)***

Show that, modulo 47, $F_n^2 - 1$ is a perfect square if $n$ is not divisible by 16.

***Solution by L. A. G. Dresel, Reading, England***

We note that for $n = 1$ to 8, $(F_n)^2 - 1$ is given successively by 0, 0, 3, 8, 24, 63, 168, 440, and that modulo 47 these numbers are congruent to the squares of 0, 0, 12, 14, 20, 4, 11, and 8, respectively. The identity (15b) of [1] gives $F_{8+m} - (-1)^m F_{8-m} = F_m L_8$, and since $L_8 = 47$ this gives $(F_{8+m})^2 \equiv (F_{8-m})^2 \pmod{47}$. Hence, modulo 47, we have $(F_n)^2 -$ as a perfect square for $n = 1$ to 15. Finally, the identity (15a) of [1] gives $F_{16+m} + (-1)^m F_{16-m} = L_m F_{16} \equiv 0 \pmod{47}$, since $F_{16} = F_8 L_8$. This completes the proof that $(F_n)^2 -$ is a perfect square modulo 47 if $n$ is not divisible by 16. When $n$ is divisible by 16, $F_n \equiv 0 \pmod{47}$, and $-1$ is not a perfect square modulo 47.

**Reference**

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester: Ellis Horwood Ltd., 1989.

***Also solved by Richard André-Jeannin, Brian D. Beasley, Paul S. Bruckman, H.-J. Seiffert, and the proposer.***

## A Sum of Product of Fibonacci Numbers That Is Identically Zero

**B-893** ***Proposed by Aloysius Dorp, Brooklyn, NY***
***(Vol. 38, no. 1, February 2000)***

Find integers $a$, $b$, $c$, and $d$ so that

$$F_x F_y F_z + aF_{x+1}F_{y+1}F_{z+1} + bF_{x+2}F_{y+2}F_{z+2} + cF_{x+3}F_{y+3}F_{z+3} + dF_{x+4}F_{y+4}F_{z+4} = 0$$

is true for all $x$, $y$, and $z$.

***Solution by L. A. G. Dresel, Reading, England***

Let

$$T(x, y, z) = F_x F_y F_z + aF_{x+1}F_{y+1}F_{z+1} + bF_{x+2}F_{y+2}F_{z+2} + cF_{x+3}F_{y+3}F_{z+3} + dF_{x+4}F_{y+4}F_{z+4},$$

giving

$$T(x, y, -4) = -3F_x F_y + 2aF_{x+1}F_{y+1} - bF_{x+2}F_{y+2} + cF_{x+3}F_{y+3}$$

and

$$T(x, y, -3) = 2F_xF_y - aF_{x+1}F_{y+1} + bF_{x+2}F_{y+2} + dF_{x+4}F_{y+4}.$$

But, from the recurrences for $F_x$ and $F_y$, we obtain

$$F_{x+3}F_{y+3} = (F_{x+2} + F_{x+1})(F_{y+2} + F_{y+1})$$

and

$$F_xF_y = (F_{x+2} - F_{x+1})(F_{y+2} - F_{y+1}).$$

Adding these together gives the identity

$$F_{x+3}F_{y+3} - 2F_{x+2}F_{y+2} - 2F_{x+1}F_{y+1} + F_xF_y = 0.$$

Denoting the left side of this by $D(x, y)$, we have $D(x, y) = 0$ for all $x$ and $y$. We can now choose values for $a$, $b$, and $c$ to make $T(x, y, -4) = -3D(x, y)$ identically, namely $a = 3$, $b = -6$, $c = -3$. If, in addition, we choose $d = 1$, we find that $T(x, y, -3) = 2D(x, y) + D(x+1, y+1)$. It follows that with these values of $a$, $b$, $c$, and $d$ we have $T(x, y, -4) = 0$ and $T(x, y, -3) = 0$. Furthermore, since the recurrence for $F_z$ gives

$$T(x, y, z+2) = T(x, y, z+1) + T(x, y, z),$$

we can prove by induction on $z$ that $T(x, y, z) = 0$ for all $x$, $y$, and $z$. Hence, we have the identity

$$F_xF_yF_z + 3F_{x+1}F_{y+1}F_{z+1} - 6F_{x+2}F_{y+2}F_{z+2} - 3F_{x+3}F_{y+3}F_{z+3} + F_{x+4}F_{y+4}F_{z+4} = 0.$$

*Paul Bruckman noted that the coefficients 3, $-6$, $-3$, and 1 correspond to the coefficients appearing in the recurrence relation satisfied by the cubes of the Fibonacci numbers. This is also seen by setting $x = y = z$ in the equation.*

***Also solved by Brian D. Beasley, Paul S. Bruckman, Hradec Královè, H.-J. Seiffert, and the proposer.***

**Addendum:** We wish to belatedly acknowledge solutions from Paul S. Bruckman to Problems B-884, B-885, B-887, and B-888, and from H.-J. Seiffert to Problems B-878, B-879, B-880, B-881, and B-882.

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