ON THE k-ARY CONVOLUTION OF ARITHMETICAL FUNCTIONS

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1. INTRODUCTION

A divisor d of n is said to be a unitary divisor of n if the greatest common divisor of d and n/d is 1 (see [4], [9]), and a divisor d of n is said to be a biunitary divisor of n if the greatest common unitary divisor of d and n/d is 1 (see [11], [12]). It is easy to see that the unitary divisors of a prime power p^a $(a \geq 1)$ are 1 and p^a, and the biunitary divisors of p^a $(a \geq 1)$ are $1, p, p^2, \ldots, p^a$, except for $p^{a/2}$ when a is even. Cohen [5] extends the above notions inductively.

Definition 1.1: If $d \mid n$, then d is a 0-ary divisor of n. For $k \geq 1$, a divisor d of n is a k-ary divisor of n if the greatest common $(k-1)$-ary divisor of d and n/d is 1.

Remark: Different extensions of the concept of a unitary divisor have been developed by Suryanarayana [10] (who also used the term k-ary divisor) and Alladi [1]. We do not consider these extensions here.

We write $d \mid_k n$ to mean that d is a k-ary divisor of n, and $(m, n)_k$ to stand for the greatest common k-ary divisor of m and n. Thus, for $k \geq 1$, $d \mid_k n$ if and only if $d \mid n$ and $(d, n/d)_{k-1} = 1$ with the convention that $(d, n/d)_0 = (d, n/d)$. In particular, $d \mid_1 n$ (resp. $d \mid_2 n$) means that d is a unitary (resp. biunitary) divisor of n.

Definition 1.2: We say that p^b is an infinitary divisor of p^a $(a \geq 1)$ (written as $p^b \mid_\infty p^a$) if $p^b \mid_{a-1} p^a$. In addition, 1 is the only infinitary divisor of 1. Further, $d \mid_\infty n$ if $p^{d(p)} \mid_\infty p^{n(p)}$ for all primes p, where $d = \prod_p p^{d(p)}$ and $n = \prod_p p^{n(p)}$ are the canonical forms of d and n.

The justification for Definition 1.2 is that, for $k \geq a-1 \geq 0$, $p^b \mid_k p^a \Leftrightarrow p^b \mid_{a-1} p^a$ (see [5]). Thus, for $k \geq a-1 \geq 0$,

$$p^b \mid_k p^a \Leftrightarrow p^b \mid_\infty p^a. \tag{1.1}$$

This means that, for $a = 0, 1, 2, \ldots, k+1$, the k-ary divisors of p^a are the same as the infinitary divisors of p^a. For example, for $a = 0, 1, 2, \ldots, 101$, the 100-ary divisors of p^a are the infinitary divisors of p^a.

Cohen and Hagis ([5], [6], [7]) give an elegant method for determining infinitary divisors. Let $I = \{p^{2^\alpha} \mid p$ is a prime, α is a nonnegative integer$\}$. It follows from the fundamental theorem of arithmetic and the binary representation that every n (> 1) can be written in exactly one way (except for the order of factors) as the product of distinct elements of I. Each element of I in this product is called an I-component of n. Cohen and Hagis ([5], [6], [7]) also note that $d \mid_\infty n$ if and only if every I-component of d is also an I-component of n with the convention that $1 \mid_\infty n$ for all n. For example, if $n = 2^3 3^5 = 2 \cdot 2^2 \cdot 3 \cdot 3^4$, then the I-components of n are $2, 2^2, 3, 3^4$. Note that this method makes it possible to compute the k-ary divisors of the prime powers $1, p, p^2, \ldots, p^{k+1}$. A general formula for the k-ary divisors of p^a for $a \geq k+2$ is not known.

The concept of divisor is related to the Dirichlet convolution of arithmetical functions. The concepts of unitary and biunitary divisor lead to the unitary and biunitary convolution. This suggests we define the k-ary convolution of arithmetical functions f and g as

$$(f *_k g)(n) = \sum_{d \mid_k n} f(d)g(n/d)$$

for $k \geq 0$. In particular, the 0-ary, 1-ary, and 2-ary convolution is the Dirichlet, unitary, and biunitary convolution, respectively.

The purpose of this paper is to represent the basic algebraic properties of the k-ary convolution and to study the Möbius function under the k-ary convolution.

2. BASIC PROPERTIES OF THE k-ARY CONVOLUTION

In this section we represent the basic algebraic properties of the k-ary convolution. Particular attention is paid to multiplicative functions. An arithmetical function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, and an arithmetical function f is said to be completely multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all m and n. Cohen and Hagis [6] say that an arithmetical function f is I-multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m, n)_\infty = 1$, where $(m, n)_\infty$ is the greatest common infinitary divisor of m and n. It is easy to see that

$$\begin{aligned} f \text{ is completely multiplicative } &\Rightarrow f \text{ is } I\text{-multiplicative} \\ &\Rightarrow f \text{ is multiplicative.} \end{aligned} \tag{2.1}$$

Theorem 2.1: Let $k \geq 0$.

1) The k-ary convolution is commutative.

2) The function δ serves as the identity under the k-ary convolution, where $\delta(1) = 1$ and $\delta(n) = 0$ for $n \geq 2$.

3) An arithmetical function f possesses an inverse under the k-ary convolution if and only if $f(1) \neq 0$. The inverse $(f^{-1})_k$ is given recursively as $(f^{-1})_k(1) = 1/f(1)$ and, for $n \geq 2$,

$$(f^{-1})_k(n) = \frac{-1}{f(1)} \sum_{\substack{d \mid_k n \\ d>1}} f(d)(f^{-1})_k(n/d). \tag{2.2}$$

4) The k-ary convolution preserves multiplicativity, that is, if f and g are multiplicative, so is their k-ary convolution.

5) If f is multiplicative, so is $(f^{-1})_k$.

Proof: Theorem 2.1 can be proved by adopting the standard argument (see, e.g., [2], [9]). As part 5 is needed later, we present the details of the proof of part 5. Assume that $(m, n) = 1$. If $mn = 1$, then $(f^{-1})_k(mn) = 1 = (f^{-1})_k(m)(f^{-1})_k(n)$. Assume that $mn \neq 1$ and that $(f^{-1})_k(m'n') = (f^{-1})_k(m')(f^{-1})_k(n')$ whenever $(m', n') = 1$ and $m'n' < mn$. If $m = 1$ or $n = 1$, then $(f^{-1})_k(mn) = (f^{-1})_k(m)(f^{-1})_k(n)$. Assume that $m, n \neq 1$. With the aid of (2.2), we obtain

$$
\begin{aligned}
(f^{-1})_k(mn) &= -\sum_{\substack{d\,|_k\,mn\\ d>1}} f(d)(f^{-1})_k(mn/d) = -\sum_{\substack{d_1\,|_k\,m\\ d_2\,|_k\,n\\ d_1d_2>1}} f(d_1d_2)(f^{-1})_k((m/d_1)(n/d_2))\\
&= -\sum_{\substack{d_1\,|_k\,m\\ d_2\,|_k\,n\\ d_1d_2>1}} f(d_1)f(d_2)(f^{-1})_k(m/d_1)(f^{-1})_k(n/d_2)\\
&= -(f^{-1})_k(m)\sum_{\substack{d_2\,|_k\,n\\ d_2>1}} f(d_2)(f^{-1})_k(n/d_2) - (f^{-1})_k(n)\sum_{\substack{d_1\,|_k\,m\\ d_1>1}} f(d_1)(f^{-1})_k(m/d_1)\\
&\qquad -\sum_{\substack{d_1\,|_k\,m\\ d_1>1}} f(d_1)(f^{-1})_k(m/d_1)\sum_{\substack{d_2\,|_k\,n\\ d_2>1}} f(d_2)(f^{-1})_k(n/d_2)\\
&= (f^{-1})_k(m)(f^{-1})_k(n) + (f^{-1})_k(m)(f^{-1})_k(n) - (f^{-1})_k(m)(f^{-1})_k(n)\\
&= (f^{-1})_k(m)(f^{-1})_k(n).
\end{aligned}
$$

This completes the proof. □

Remark: The k-ary convolution is not associative in general. For example, the biunitary convolution is not associative (see [8]).

The infinitary convolution [6] of arithmetical functions f and g is defined as

$$(f *_\infty g)(n) = \sum_{d\,|_\infty\,n} f(d)g(n/d).$$

The infinitary convolution possesses the properties given in Theorem 2.1. In addition, it is associative and possesses basic properties with respect to I-multiplicative functions. We present these results in the following theorem.

Theorem 2.2:

1) The infinitary convolution is associative.

2) The infinitary convolution is commutative.

3) The function δ serves as the identity under the infinitary convolution, where $\delta(1) = 1$ and $\delta(n) = 0$ for $n \geq 2$.

4) An arithmetical function f possesses an inverse under the infinitary convolution if and only if $f(1) \neq 0$. The inverse $(f^{-1})_\infty$ is given recursively as $(f^{-1})_\infty(1) = 1/f(1)$ and, for $n \geq 2$,

$$(f^{-1})_\infty(n) = \frac{-1}{f(1)} \sum_{\substack{d\,|_\infty\,n\\ d>1}} f(d)(f^{-1})_\infty(n/d). \tag{2.3}$$

5) The infinitary convolution preserves multiplicativity.

6) If f is multiplicative, so is $(f^{-1})_\infty$.

7) The infinitary convolution preserves I-multiplicativity.

8) If f is I-multiplicative, so is $(f^{-1})_\infty$.

Theorem 2.2 is given in Cohen and Hagis [6] except for equation (2.3) and parts 5 and 6. Cohen and Hagis [6] do not prove their results. We do not prove these results either, since the standard argument (see, e.g., [2], [9]) can be applied.

Remark: It is easy to see that the k-ary convolution for all k and the infinitary convolution do not preserve complete multiplicativity.

Remark: Theorem 2.2 shows that I-multiplicative functions possess two basic properties under the infinitary convolution. This leads us to propose the following unsolved research problem. Define k-ary multiplicative functions so that they possess basic properties under the k-ary convolution.

3. THE k-ARY MÖBIUS FUNCTION

We define the k-ary Möbius function μ_k as the inverse of the constant function 1, denoted by ζ, under the k-ary convolution. In particular, μ_0 is the classical number-theoretic Möbius function and μ_1 is the unitary Möbius function (see [4], [9]). Since ζ is a multiplicative function, so is μ_k. Therefore, μ_k is completely determined by its values at prime powers. The values of μ_k at prime powers are obtained recursively as $\mu_k(1)=1$ and, for $a \geq 1$,

$$\mu_k(p^a) = -\sum_{\substack{p^b \mid_k p^a \\ 0 \leq b < a}} \mu_k(p^b). \tag{3.1}$$

A general explicit formula for μ_k is not known.

We define the infinitary Möbius function μ_∞ as the inverse of the function ζ under the infinitary convolution. An explicit formula for μ_∞ is known. Let $s_2(a)$ denote the number of nonzero terms in the binary representation of a with the convention that $s_2(0)=0$, and let $J(n)$ denote the arithmetical function defined as $J(1)=0$ and, for $n \geq 2$, $J(n) = \sum_p s_2(n(p))$, where $n = \prod_p p^{n(p)}$ is the canonical form of n. Note that $J(n)$ is the number of I-components of n. Cohen and Hagis [6] show that

$$\mu_\infty(n) = (-1)^{J(n)}. \tag{3.2}$$

It follows from (1.1) that

$$\mu_k(p^a) = \mu_\infty(p^a) \quad \text{for } a = 0, 1, 2, \ldots, k+1. \tag{3.3}$$

Therefore, in a sense, μ_k comes closer to μ_∞ as k increases.

It is interesting that

$$\mu_2 = \mu_\infty. \tag{3.4}$$

This is a consequence of Theorem 3.1 given below and equation (3.2).

Theorem 3.1: If f is completely multiplicative, then

$$(f^{-1})_2(n) = (-1)^{J(n)} f(n). \tag{3.5}$$

Proof: Since both sides of (3.5) are multiplicative functions in n, we may confine ourselves to prime powers p^a. By (2.2), and knowing the biunitary divisors of p^a, we have $(f^{-1})_2(1)=1$ and, for $a \geq 1$,

$$\begin{cases} \sum_{i=0}^{a} (f^{-1})_2(p^i)f(p^{a-i}) = 0 & \text{if } a \text{ is odd,} \\ \sum_{i=0}^{a} (f^{-1})_2(p^i)f(p^{a-i}) - (f^{-1})_2(p^{a/2})f(p^{a/2}) = 0 & \text{if } a \text{ is even.} \end{cases}$$

Therefore, for $a \geq 0$,

$$\begin{cases} \sum_{i=0}^{2a+1} (f^{-1})_2(p^i)f(p^{2a+1-i}) = 0, \\ \sum_{i=0}^{2a} (f^{-1})_2(p^i)f(p^{2a-i}) - (f^{-1})_2(p^a)f(p^a) = 0. \end{cases}$$

This shows that the function $(f^{-1})_2$ at prime powers is completely determined by the recurrence relation

$$\begin{cases} (f^{-1})_2(p^{2a+1}) + f(p^{a+1})(f^{-1})_2(p^a) = 0, \\ (f^{-1})_2(p^{2a+2}) - f(p^{a+1})(f^{-1})_2(p^{a+1}) = 0, \end{cases}$$

for $a \geq 0$, with the initial condition $(f^{-1})_2(1) = 1$.

We show that the function $g(n) = (-1)^{J(n)} f(n)$ satisfies the same recurrence relation at prime powers. In fact,

$$\begin{aligned} g(p^{2a+1}) + f(p^{a+1})g(p^a) &= (-1)^{s_2(2a+1)} f(p^{2a+1}) + f(p^{a+1})(-1)^{s_2(a)} f(p^a) \\ &= (-1)^{s_2(a)+1} f(p)^{2a+1} + f(p)^{a+1}(-1)^{s_2(a)} f(p)^a = 0 \end{aligned}$$

and

$$\begin{aligned} g(p^{2a+2}) - f(p^{a+1})g(p^{a+1}) &= (-1)^{s_2(2a+2)} f(p^{2a+2}) - f(p^{a+1})(-1)^{s_2(a+1)} f(p^{a+1}) \\ &= (-1)^{s_2(a+1)} f(p)^{2a+2} - f(p)^{a+1}(-1)^{s_2(a+1)} f(p)^{a+1} = 0, \end{aligned}$$

for $a \geq 0$, with initial condition $g(1) = 1$. This completes the proof. □

Remark: The idea for the recurrence relation in the proof of Theorem 3.1 is developed from [3].

Cohen and Hagis [6] show that, if f is I-multiplicative, then

$$(f^{-1})_\infty(n) = (-1)^{J(n)} f(n).$$

On the basis of equations (2.1) and (3.5), we see that, if f is completely multiplicative, then

$$(f^{-1})_2 = (f^{-1})_\infty. \tag{3.6}$$

Since the function ζ is completely multiplicative, we obtain equation (3.4).

Remark: It is an open question whether (3.6) holds for all I-multiplicative functions f.

It is known [5] that the 3-ary divisors of p^a are 1 and p^a, except for the cases $a = 3$ and $a = 6$. The 3-ary divisors of p^3 are $1, p, p^2, p^3$, and the 3-ary divisors of p^6 are $1, p^2, p^4, p^6$. Using this result and (3.1), we conclude that

$$\mu_3(p^a) = \begin{cases} 1 & \text{if } a = 0, 3, 6, \\ -1 & \text{otherwise.} \end{cases}$$

Thus, in the case $k=3$, we have $\mu_k(p^a)=\mu_\infty(p^a)$ for $a=0,1,2,\dots,k+1$ (cf. (3.3)), but $\mu_k(p^{k+2})=-\mu_\infty(p^{k+2})$ or $\mu_3(p^5)=-1=-\mu_\infty(p^5)$. Further evaluations of μ_k for small values of k could be derived using the results on k-ary divisors given in [5].

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