CONVOLUTION SUMMATIONS FOR PELL AND PELL-LUCAS NUMBERS

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1. RATIONALE

Pell and Pell-Lucas Convolution Numbers

Pell and Pell-Lucas polynomials $P_n(x)$ and $Q_n(x)$, respectively, were investigated in some detail in [3], which was followed up with a study of the properties [4] of the m^{th} convolution polynomials $P_n^{(m)}(x)$ and $Q_n^{(m)}(x)$.

These convolution polynomials may be defined [4] by generating functions, thus:

$$\sum_{n=0}^{\infty} P_{n+1}^{(m)}(x)y^n = (1-2xy-y^2)^{-(m+1)} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} Q_{n+1}^{(m)}(x)y^n = \left(\frac{2x+2y}{1-2xy-y^2}\right)^{m+1} \tag{1.2}$$

Putting $x=1$ yields the m^{th} convolution Pell and Pell-Lucas numbers $P_n^{(m)}(1)$ and $Q_n^{(m)}(1)$, respectively. Furthermore, if also $m=0$, then we have the Pell numbers $P_n^{(0)}(1)=P_n$ and the Pell-Lucas numbers $Q_n^{(0)}(1)=Q_n$.

Recurrence relations are given in (2.1) and (2.2) for $P_n^{(m)}$, and in (3.1) with (3.2) for $Q_n^{(m)}$ ($m \geq 1$ in both cases). Further specific work on P_n and Q_n was related to Morgan-Voyce numbers in [2].

Morgan-Voyce and Quasi Morgan-Voyce Polynomials

Morgan-Voyce polynomials $X_n(x)=B_n(x)$, $b_n(x)$, $C_n(x)$, and $c_n(x)$, and the four associated quasi Morgan-Voyce polynomials $Y_n(x)=\mathcal{B}_n(x)$, $\mathbf{b}_n(x)$, $\mathcal{C}_n(x)$, and $\mathbf{c}_n(x)$ are defined [1], [2] recursively by

$$X_{n+2}(x)=X_{n+1}(x)-3X_n(x),\ X_0(x)=a,\ X_1(x)=b, \tag{1.3}$$

and

$$Y_{n+2}(x)=Y_{n+1}(x)+3Y_n(x),\ Y_0(x)=a,\ Y_1(x)=b, \tag{1.4}$$

(a, b integers), in accordance with the following tabulation:

$X_n(x)$	a	b	$Y_n(x)$
$B_n(x)$	0	1	$\mathcal{B}_{n+1}(x)$
$b_n(x)$	1	1	$\mathbf{b}_{n+1}(x)$
$C_n(x)$	2	$2+x$	$\mathcal{C}_n(x)$
$c_n(x)$	-1	1	$\mathbf{c}_{n+1}(x)$

(1.5)

Only $\mathcal{B}_n(x)$ is required in this paper.

Our Challenge

Yet remaining for attention are some additional data to be obtained for $P_n^{(m)}(x)$ in Section 2, to be complemented by a corresponding, and slightly more thorough, analysis of properties of $Q_n^{(m)}(x)$ in Section 3.

In particular, our study of the row sums and column sums of $P_n^{(m)}$ and $Q_n^{(m)}$, as well as the rising diagonal sums $\sum_{m=1}^{n} P_m^{(n-m)}$ and $\sum_{m=1}^{n} Q_m^{(n-m)}$ will reveal some pleasing features.

For ease of reference and calculation, the short table of Pell number convolutions $P_n^{(m)}(1)$ which appeared in [4] will necessarily have to be repeated here as Table 1. Furthermore, a new table for Pell-Lucas number convolutions $Q_n^{(m)}(1)$, not previously recorded, will have to be incorporated as Table 2. Extensions of Tables 1 and 2 may be effected by employing the recurrence relations (2.1) and (3.1).

2. NEW PROPERTIES OF PELL CONVOLUTIONS

Prompted by an observation made by a colleague at the Rochester, New York State, meeting of the Fibonacci Association (July 1998)—an observation actually covered in [2]—we begin an investigation of certain summation properties of the Pell convolutions (Table 1).

Crucial to our presentation is the *recurrence relation* [4] for Pell convolutions,

$$P_n^{(m)} = 2P_{n-1}^{(m)} + P_{n-2}^{(m)} + P_n^{(m-1)} \quad (m \geq 1), \tag{2.1}$$

with

$$P_0^{(m)} = 0. \tag{2.2}$$

An abbreviated table for these convolutions, given in [2] and [4], is repeated here for the reader's convenience.

TABLE 1. Pell Convolution Numbers $P_n^{(m)}$

$n \backslash m$	0	1	2	3	4
1	1	1	1	1	1
2	2	4	6	8	10
3	5	14	27	44	65
4	12	44	104	200	340
5	29	131	366	810	1555

When required for formal algebraic purposes, values of $P_n^{(m)}$ could be extended for negative n in (2.1).

Basically, our concern is with **three** summation formulas, namely, those for rows, columns, and rising diagonals in Table 1.

Row Sums

Theorem 1: $\displaystyle\sum_{k=0}^{m} P_n^{(k)} = \frac{1}{2}\left\{P_{n+1}^{(m)} - \sum_{k=0}^{m} P_{n-1}^{(k)}\right\}$ (**n fixed**).

Proof: Write out (2.1) for successive values of m $(=0,1,\ldots,k)$ with n **fixed**. Add (the columns) to obtain

$$\sum_{k=0}^{m} P_n^{(k)} = 2\sum_{k=0}^{m} P_{n-1}^{(k)} + \sum_{k=0}^{m} P_{n-2}^{(k)} + \sum_{k=0}^{m-1} P_n^{(k)},$$

$$P_n^{(m)} + \sum_{k=0}^{m-1} P_n^{(k)} = 2\sum_{k=0}^{m} P_{n-1}^{(k)} + \sum_{k=0}^{m} P_{n-2}^{(k)} + \sum_{k=0}^{m-1} P_n^{(k)},$$

whence the result enunciated for k follows on replacing n by $n+1$.

***Example* ($n = 3$, $m = 4$):** Theorem $1 \to 2\times 155 = 340 - 30 (= 310)$.

Column Sums

Theorem 2: $\displaystyle\sum_{i=1}^{n} P_i^{(m)} = \frac{1}{2}\left\{P_{n+1}^{(m)} + P_n^{(m)} - \sum_{i=1}^{n+1} P_i^{(m-1)}\right\}$ (**m fixed**).

Proof: Proceed as in Theorem 1 (**m fixed**). Quickly it follows that

$$\begin{aligned} 2\sum_{i=1}^{n} P_i^{(m)} &= P_{n+2}^{(m)} - P_{n+1}^{(m)} - \sum_{i=1}^{n+2} P_i^{(m-1)} \\ &= P_{n+1}^{(m)} + P_n^{(m)} + P_{n+2}^{(m-1)} - \sum_{i=1}^{n+2} P_i^{(m-1)} \quad \text{by (2.1)} \\ &= P_{n+1}^{(m)} + P_n^{(m)} - \sum_{i=1}^{n+1} P_i^{(m-1)}. \end{aligned}$$

Hence, the theorem is demonstrated.

***Example* ($m = 3$, $n = 4$):** Theorem $2 \to 253 = \frac{1}{2}\{810 + 200 - 504\}$.

Note: For $m = 0$ (excluded from Theorem 2), we have [3, (2.11)] where $x = 1$,

$$\sum_{i=0}^{n} P_i = \frac{1}{2}\{P_{n+1} + P_n - 1\}. \tag{2.3}$$

Rising Diagonal Sums

Upward slanting (i.e., rising) diagonals are to be imagined in the mind's eye in Table 1. Accordingly, we seek $\sum_{m=1}^{n} P_m^{(n-m)}$. Specifically, these convolution number sums $\sum_{m=1}^{n} P_m^{(n-m)}$ turn out empirically to be the sequence

$$(0), 1, 3, 10, 33, 109, 360, \ldots = F_n(3), \tag{2.4}$$

where $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ $(F_0(x) = 0, F_1(x) = 1)$ are the Fibonacci polynomials.

Why is this so?

Theorem 3: $\displaystyle\sum_{m=1}^{n} P_m^{(n-m)} = F_n(3)$.

Proof (by induction): For small values $n = 1, 2, 3, 4$ (say), the validity of the theorem is clearly verifiable. Suppose it is true for $n = N$ (fixed). That is, assume

$$P_1^{(N-1)} + P_2^{(N-2)} + P_3^{(N-3)} + \cdots + P_{N-2}^{(2)} + P_{N-1}^{(1)} + P_N^{(0)} = F_n(3). \tag{A}$$

Apply the recurrence relation (2.1) repeatedly for $m = 1, 2, \ldots, N+1$. Arrange the summations in three columns, in accordance with (2.1). Then

$$\begin{aligned}\sum_{m=1}^{N+1} P_m^{(N+1-m)} &= P_1^{(N)} + P_2^{(N-1)} + P_3^{(N-2)} + \cdots + P_{N-1}^{(2)} + P_N^{(1)} + P_{N+1}^{(0)} \\ &= 2F_N(3) + F_{N-1}(3) + F_N(3) \quad \text{by (2.1) and (A)} \\ &= 3F_N(3) + F_{N-1}(3) \\ &= F_{N+1}(3) \text{ by the definition of } F_n(x) \text{ above.}\end{aligned}$$

Hence, the theorem is valid for $n = N+1$.

Consequently, Theorem 3 has been demonstrated for all n.

Indeed [2]

$$F_n(3) = \mathcal{B}_n(1) \equiv \mathcal{B}_n, \tag{2.5}$$

where $\mathcal{B}_n$ are quasi Morgan-Voyce numbers (of one kind) formed from the quasi Morgan-Voyce polynomials $\mathcal{B}_n(x)$ when $x = 1$.

Now the *Binet form* for these quasi Morgan-Voyce numbers is [2]

$$\mathcal{B}_n = (\alpha^n - \beta^n) / \Delta, \tag{2.6}$$

where α, β are the roots of the characteristic quasi Morgan-Voyce equation

$$\lambda^2 - 3\lambda - 1 = 0, \tag{2.7}$$

whence

$$\alpha = \frac{3+\sqrt{13}}{2},\ \beta = \frac{3-\sqrt{13}}{2},\ \alpha\beta = -1,\ \alpha+\beta = 3,\ \alpha-\beta = \Delta = \sqrt{13}. \tag{2.8}$$

Combining these ideas, we deduce that

Theorem 3a: $\sum_{m=1}^{n} P_m^{(n-m)} = \mathcal{B}_n = \dfrac{\alpha^n - \beta^n}{\Delta}$, where α, β, Δ are defined in (2.8).

Example ($n = 5$): $\sum_{m=1}^{5} P_m^{(5-m)} \equiv \dfrac{\alpha^5 - \beta^5}{\alpha - \beta} = 109 = \mathcal{B}_5$.

As an extension, the sum of the $\mathcal{B}_n$ (i.e., the sum of the sums of the rising diagonal convolutions) reduces, after algebraic maneuvering, to

Theorem 4: $\sum_{n=1}^{k} \mathcal{B}_n = \dfrac{1}{3}(\mathcal{B}_{k+1} + \mathcal{B}_k - 1)$.

Example ($k = 5$): Theorem $4 \to 156 = \frac{1}{3}(360 + 109 - 1)$.

Properties of the quasi Morgan-Voyce numbers $\mathcal{B}_n$ which are well documented in [2] may, because of Theorem 3a, be conceived in terms of sums of rising diagonal Pell convolutions. Recall that $\mathcal{B}_n = \mathcal{B}_n(x)$ when $x = 1$.

One might compare the forms on the right-hand side in Theorem 4 and equation (2.3).

3. NEW PROPERTIES OF PELL-LUCAS CONVOLUTIONS

Recurrence Relation

Coming now to the Pell-Lucas convolution polynomials $Q_n^{(m)}$, we must first discover their recurrence relation, a fundamental requirement which was not incorporated into [4].

Ordinarily, one might reasonably anticipate that the form of this recurrence relation would closely resemble that in (2.1). However, there is an unexpected scorpion-like twist to the tail of this formula.

Empirical evidence enables us to spot the following recurrence relation, cf. (2.1),

$$Q_n^{(m)} = 2Q_{n-1}^{(m)} + Q_{n-2}^{(m)} + 2(Q_n^{(m-1)} + Q_{n-1}^{(m-1)}) \quad (m \geq 1) \tag{3.1}$$

with

$$Q_0^{(m)} = 2. \tag{3.2}$$

Substituting $m = 1$ in (3.1) reduces the bracketed "tail" to $4P_n$.

On the basis of (3.1) and (3.2), we can construct a shortened convolution array for $Q_n^{(m)}$ (Table 2). Recall that a few simple values ($m = 1, 2$; $n = 1, 2, 3, 4, 5$) could readily have been calculated from the data in the table on page 68 in [4].

TABLE 2. Pell-Lucas Convolution Numbers $Q_n^{(m)}$

$n \backslash m$	0	1	2	3	4
1	2	4	8	16	32
2	6	24	72	192	480
3	14	92	384	1312	4004
4	34	304	1632	6848	24810
5	82	932	6120	30512	128344

Extension Example: $Q_6^{(1)} = 2Q_5^{(1)} + Q_4^{(1)} + 2(Q_6 + Q_5) = 1864 + 304 + 2(198 + 82) = 2728.$

Paralleling the triad of Theorems 1-3 in Section 2, we now explore the new territory for $Q_n^{(m)}$. Not unexpectedly, the forms of the corresponding enunciations are not quite so pleasing to the eye, because of (3.1).

Row Sums

Theorem 5: $\displaystyle\sum_{k=0}^{m} Q_n^{(k)} = Q_{n-1}^{(m+1)} - 2Q_{n-1}^{(m+1)} - 4\sum_{k=0}^{m} Q_{n-1}^{(k)} - 2(2^{m+1} - 1)$ (**n fixed**).

Proof: Proceed as for Theorem 1.

Example (m = 3, n = 3): $\displaystyle\sum_{k=0}^{3} Q_3^{(k)} = 4004 - 964 - 1176 - 62 (= 1802).$

Column Sums

Aesthetically, we are blessed with no more joy here than we were in Theorem 5.

Theorem 6: $\displaystyle\sum_{k=2}^{n-2} Q_k^{(m)} = \frac{1}{2}\{Q_n^{(m)} - Q_{n-1}^{(m)}\} - 2\sum_{k=2}^{n-1} Q_k^{(m-1)} - Q_n^{(m-1)} - 2^{m+2}\Bigg\}$ **m fixed**, $n \geq 2$.

Proof: As for Theorem 2.

Example (m = 2, n = 5): $456 = \frac{1}{2}\{6120 - 1632\} - 840 - 932 - 16.$

The requirements of realism necessitate the lower summation bound to be at $k = 2$. This is because $k = 0$ and $k = 1$, from (3.1), will yield terms $Q_0^{(m)}$ and $Q_{-1}^{(m)}$ which do not exist in Table 2.

Rising Diagonal Sums

Upward slanting (rising) diagonal sums are of the form $\sum_{m=1}^{n} Q_m^{(n-m)}$. Denote this by $\mathcal{Q}_n$ so that $\mathcal{Q}_1 = 2$. Then Table 2 reveals that

$$\{\mathcal{Q}_n\} = 2, 10, 46, 214, 994, 4618, \ldots, \tag{3.3}$$

whence one can spot the *recurrence relation*

$$\mathcal{Q}_{n+2} = 4\mathcal{Q}_{n+1} + 3\mathcal{Q}_n. \tag{3.4}$$

What can we know about this new sequence? Elementary procedures enable us to establish the relation

$$\mathcal{Q}_n = Z_n + Z_{n-1} \tag{3.5}$$

where the *Binet form* for Z_n is

$$Z_n = \frac{2}{\Delta_1}(\gamma^n - \delta^n), \tag{3.6}$$

in which γ, δ are the roots of the characteristic equation for (3.4), namely,

$$t^2 - 4t - 3 = 0; \tag{3.7}$$

so that

$$\gamma + \delta = 4,\ \gamma\delta = -3,\ \gamma - \delta = 2\sqrt{7} = \Delta_1. \tag{3.8}$$

Consequently, we have $(Z_0 = 0)$

$$\{Z_n\} = 2, 8, 38, 176, 818, \ldots, \tag{3.9}$$

with the same form of the recurrence relation for Z_n as that for $\mathcal{Q}_n$, i.e.,

$$Z_{n+2} = 4Z_{n+1} + 3Z_n. \tag{3.10}$$

Since $\mathcal{Q}_n$ is a composite of two Z-numbers, it is simpler to concentrate our energies on Z_n.

Generating Functions

One may readily obtain the generating function for the Z-numbers, to wit,

$$\sum_{k=1}^{\infty} Z_k x^k = 2(1 - 4x - 3x^2)^{-1}, \tag{3.11}$$

thence (3.5) engenders

$$\sum_{k=1}^{\infty} \mathcal{Q}_n x^k = (2 + 2x)(1 - 4x - 3x^2)^{-1}. \tag{3.12}$$

Summations

The Binet form (3.6) leads to

$$\sum_{k=1}^{n} Z_k = \frac{1}{6}\{Z_{n+1} + 3Z_n - 2\} \tag{3.13}$$

which, by (3.5) with (3.8), produces

$$\sum_{k=1}^{n} \mathcal{Q}_k = \frac{1}{3}(Z_{n+1} - 2). \tag{3.14}$$

Example: $\sum_{k=1}^{5} \mathcal{Q}_k = \frac{1}{3}(3800-2) = 1266.$

Simson Formulas

Invoking the application of (3.6) with (3.8), we derive the Simson formula

$$Z_{n+1}Z_{n-1} - Z_n^2 = -4(-3)^{n-1} \tag{3.15}$$

while employing (3.5) with (3.8) yields the Simson formula

$$\mathcal{Q}_{n+1}\mathcal{Q}_{n-1} - \mathcal{Q}_n^2 = -8(-3)^{n-2}. \tag{3.16}$$

Example $(\boldsymbol{n} = \mathbf{4})$***:*** Both sides of (3.14) have the value -72.

Observe, in passing, that

$$\mathcal{Q}_{n+1} - \mathcal{Q}_n = Z_{n+1} - Z_{n-1}. \tag{3.17}$$

Limits

From (3.6) and (3.5),

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \lim_{n\to\infty} \frac{\mathcal{Q}_{n+1}}{\mathcal{Q}_n} = \gamma = 2 + \sqrt{7}\,(\approx 4.646), \tag{3.18}$$

whereas by (2.6) and (2.8),

$$\lim_{n\to\infty} \frac{\mathcal{B}_{n+1}}{\mathcal{B}_n} = \alpha = \frac{3+\sqrt{13}}{2}(\approx 3.303). \tag{3.19}$$

Merely for curiosity we record that

$$\frac{\gamma}{\alpha} \approx 1.4 \quad \text{(one decimal place)}. \tag{3.20}$$

4. END-PIECE

Though the properties of the $Q_n^{(m)}$ will, by their very nature, be necessarily more complicated than those for $P_n^{(m)}$, it is nevertheless pleasing to unearth the rather unexpected conjunction of the Z's in (3.5). While other facets of the convolution numbers $P_n^{(m)}$ and $Q_n^{(m)}$ might be pursued, it seems reasonable to halt at this stage.

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