ON POLYNOMIALS RELATED TO POWERS OF THE GENERATING FUNCTION OF CATALAN'S NUMBERS

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1. INTRODUCTION AND SUMMARY

Catalan's sequence of numbers $\{C_n\}_0^{\infty} = \{1, 1, 2, 5, 14, 42, ...\}$ (nr.1459 and A000108 of [14]) emerges in the solution of many combinatorial problems (see [2], [4], [5], and [16] for further references). The moments μ_{2k} of the normalized weight function of Chebyshev's polynomials of the second kind are given by $C_k/2^k$ (see, e.g., [3], Lemma 4.3, p. 160 for l = 0, and [17], p. II-3). This sequence also shows up in the asymptotic moments of zeros of scaled Laguerre and Hermite polynomials (see [9], eqs. (3.34) and (3.35)). The generating function $c(x) = \sum_{n=0}^{\infty} C_n x^n$ is the solution of the quadratic equation $xc^2(x) - c(x) + 1 = 0$ with c(0) = 1. Therefore, every positive integer power of c(x) can be written as

$$c^{n}(x) = p_{n-1}(x)l + q_{n-1}(x)c(x), \qquad (1)$$

with certain polynomials p_{n-1} and q_{n-1} , both of degree (n-1), in 1/x. In Section 2, they are shown to be related to Chebyshev polynomials of the second kind:

$$p_{n-1}(x) = -\left(\frac{1}{\sqrt{x}}\right)^n S_{n-2}\left(\frac{1}{\sqrt{x}}\right), \ q_{n-1}(x) = \left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}\left(\frac{1}{\sqrt{x}}\right) = -xp_n(x), \tag{2}$$

with $S_n(y) = U_n(y/2)$. Therefore, it is possible to extend the range of the power *n* to negative integers (or to real or complex numbers). Tables for the $U_n(x)$ polynomials can be found, e.g., in [1]. Because powers of a generating function correspond to convolutions of the generated number sequence, the given decomposition of $c^n(x)$ will determine convolutions of the Catalan sequence. In passing, an explicit expression for general convolutions in the form of nested sums will also be given. Contact with the works of [6], [12], [18], and [5] will be made.

Together with the known (e.g., [4], [11]) result (valid for real n),

$$c^{n}(x) = \sum_{k=0}^{\infty} C_{k}(n)x^{k}, \text{ with } C_{k}(n) = \frac{n}{n+2k} \binom{n+2k}{k} = \frac{n}{k+n} \binom{n-1+2k}{k}, \quad (3)$$

one finds, from the alternative expression (1) for positive n, two sets of identities:

$$(P1) \qquad \sum_{l=0}^{p} (-1)^{l} \binom{n+1-p+l}{p-l} C_{l} = \binom{n-p}{p}$$
(4)

for $n \in \mathbb{N}_0$, $p \in \{0, 1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$, and

(P2)
$$\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n-1-l}{l} C_{k+n-1-l} = C_k(n)$$
(5)

for $n \in \mathbb{N}$, $k \in \mathbb{N}_0$.

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For negative powers in (1), two other sets of identities result:

(P3)
$$\sum_{l=0}^{\min\left(\lfloor\frac{n-1}{2}\rfloor,k-1\right)} (-1)^{l} \binom{n-1-l}{l} C_{k-1-l} = (-1)^{k+1} \binom{n-k-1}{k-1}$$
(6)

for $n \in \mathbb{N}$, $k \in \{0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ (for k = 0, both sides are by definition zero), and

$$(P4) \qquad \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{l} \binom{n-1-l}{l} C_{k-1-l} = -C_{k}(-n) = \frac{n}{k} \binom{2k-n-1}{k-1}$$
(7)

for $n \in \mathbb{N}$, $k \in \mathbb{N}$ with $k \ge \lfloor \frac{n}{2} \rfloor + 1$. These identities can be continued for appropriate values of real n.

Another expression for the coefficients of negative powers of c(x) is

$$C_{k}(-n) = \sum_{l=1}^{\min(n,k)} (-1)^{l} {n \choose l} C_{k-l}(n)$$
(8)

for $n, k \in \mathbb{N}$, and $C_0(-n) = 1$, $C_n(0) = \delta_{n,0}$. Also, from (3), $C_k(-n) = -C_{k-n}(n)$ for $n, k \in \mathbb{N}$ with $k \ge n$.

The remainder of this paper provides proofs for the above given statements. Section 2 deals with integer (and real) powers of the generating function c(x). Convolutions of general sequences are expressed there in terms of nested sums. In Section 3 some families of integer sequences related to the polynomials $q_n(x)$ (2) evaluated for x = 1/m for m = 4, 5, ... and $(-1)^n q_n(x)$ evaluated at x = -1/m, $m \in \mathbb{N}$, are considered.

2. POWERS

The equation $xc^2(x) - c(x) + 1 = 0$ whose solution defines the generating function of Catalan's numbers if c(0) = 1 can be considered as a characteristic equation for the recursion relation

$$xr_{n+1} - r_n + r_{n-1} = 0, \ n = 0, 1, \dots,$$
(9)

with arbitrary inputs $r_{-1}(x)$ and $r_0(x)$. A basis of two linearly independent solutions is given by the Lucas-type polynomials $\{\mathcal{U}_n\}$ and $\{\mathcal{V}_n\}$ with standard inputs $\mathcal{U}_{-1} = 0$, $\mathcal{U}_0 = 1$, $(\mathcal{U}_{-2} = -x)$, and $\mathcal{V}_{-1} = 1$, $\mathcal{V}_0 = 2$, $(\mathcal{V}_1 = 1/x)$, in the Binet form

$$\mathcal{U}_{n-1}(x) = \frac{c_+^n(x) - c_-^n(x)}{c_+(x) - c_-(x)},\tag{10}$$

$$\mathcal{V}_{n}(x) = c_{+}^{n}(x) + c_{-}^{n}(x) = \frac{1}{x} (\mathcal{U}_{n-1}(x) - 2\mathcal{U}_{n-2}(x)), \tag{11}$$

with the two solutions of the characteristic equation, viz $c_{\pm}(x) := (1 \pm \sqrt{1-4x})/(2x)$. $c(x) := c_{-}(x)$ satisfies c(0) = 1, and $c_{+}(x) = 1/(xc(x))$, as well as $c_{+}(x) + c(x) = 1/x$. From the recurrence (9), it is clear that, for positive $n \neq 0$, \mathcal{U}_n is a polynomial in 1/x of degree n-1. If $c_{+}(x) - c_{-}(x) = 0$, i.e., x = 1/4, equation (10) is replaced by $\mathcal{U}_n(1/4) = 2^n(n+1)$. The second equation in (11) holds because both sides of the equation satisfy recurrence (9) and the inputs for \mathcal{V}_0 and \mathcal{V}_1 match. One may associate with the recurrence relation (9) a transfer matrix

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$$\mathbf{T}(x) = \begin{pmatrix} 1/x & -1/x \\ 1 & 0 \end{pmatrix}, \quad \det \mathbf{T}(x) = 1/x.$$
(12)

With this matrix, one can rewrite (9) as

$$\binom{r_n}{r_{n-1}} = \mathbf{T}(x) \binom{r_{n-1}}{r_{n-2}} = \mathbf{T}^n(x) \binom{r_0(x)}{r_{-1}(x)}.$$
(13)

Because $\mathbf{T}^n = \mathbf{T}\mathbf{T}^{n-1}$ with input $\mathbf{T}^1 = \mathbf{T}(x)$ given by (12), one finds from the recurrence relation (9) with $r_n = \mathcal{U}_n$ that

$$\mathbf{T}^{n}(x) = \begin{pmatrix} \mathcal{U}_{n}(x) & -\frac{1}{x} \,\mathcal{U}_{n-1}(x) \\ \mathcal{U}_{n-1}(x) & -\frac{1}{x} \,\mathcal{U}_{n-2}(x) \end{pmatrix}.$$
 (14)

Note that, for x = 1, one has $c_{\pm}(1) = (1 \pm i\sqrt{3})/2$, which are 6th roots of unity, and the related period 6 sequences are $\{\mathcal{U}_n(1)\}_{-1}^{\infty} = \{\overline{0, 1, 1, 0, -1, -1}\}$, as well as $\{\mathcal{V}_n(1)\}_0^{\infty} = \{\overline{2, 1, -1, -2, -1, 1}\}$. This follows from equations (10) and (11). It is convenient to map the recursion relation (9) to the familiar one for Chebyshev's $S_n(x) = U_n(x/2)$ polynomials of the second kind, viz

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x), \ S_{-1} = 0, \ S_0 = 1,$$
(15)

with characteristic equation $\lambda^2 - x\lambda + 1 = 0$ and solutions $\lambda_{\pm}(x) = \frac{x}{2} (1 \pm \sqrt{1 - (2/x)^2})$, satisfying $\lambda_{\pm}(x)\lambda_{-}(x) = 1$ and $\lambda_{\pm}(x) + \lambda_{-}(x) = x$. The relation to $c_{\pm}(x)$ is

$$\sqrt{x} c_{\pm}(x) = \lambda_{\pm}(1/\sqrt{x}). \tag{16}$$

The Binet form of the corresponding two independent polynomial systems is

$$S_{n-1}(x) = \frac{\lambda_{+}^{n}(x) - \lambda_{-}^{n}(x)}{\lambda_{+}(x) - \lambda_{-}(x)},$$
(17)

$$2T_n(x/2) = \lambda_+^n(x) + \lambda_-^n(x),$$
(18)

and $T_n(x/2) = (S_n(x) - S_{n-2}(x))/2$ are Chebyshev polynomials of the first kind. Tables of Chebyshev polynomials can be found in [1]. The coefficient triangles of the $S_n(x)$, $U_n(x)$, and $T_n(x)$ polynomials can also be viewed under the numbers A049310, A053117, and A053120, respectively, in the on-line database [14].

The extension to negative indices runs as follows:

$$\mathcal{U}_{-n}(x) = -x^{n-1}\mathcal{U}_{n-2}(x),\tag{19}$$

$$S_{-(n+2)}(x) = -S_n(x).$$
(20)

This follows from (10) and (17). Note that from (9), \mathcal{U}_n is for positive *n* a monic polynomial in 1/x of degree *n*, and for negative *n* in general, a nonmonic polynomial in *x* of degree $\lfloor -\frac{n}{2} \rfloor$. It is possible to extend the range of *n* to complex numbers using the Binet forms.

A connection between both systems of polynomials is made, using (10), (16), and (17), by

$$\mathcal{U}_n(x) = \left(\frac{1}{\sqrt{x}}\right)^n S_n(1/\sqrt{x}).$$
(21)

This holds for $n \in \mathbb{Z}$ in accordance with (19) and (20).

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After these preliminaries, we are ready to state the following proposition.

Proposition 1: The n^{th} power of c(x), the generating function of Catalan numbers can, for $n \in \mathbb{Z}$, be written as

$$c^{n}(x) = -\frac{1}{x} \mathcal{U}_{n-2}(x) + \mathcal{U}_{n-1}(x)c(x), \qquad (22)$$

$$= -\left(\frac{1}{\sqrt{x}}\right)^{n} S_{n-2}(1/\sqrt{x}) + \left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}(1/\sqrt{x})c(x).$$
(23)

Proof: Due to $c^2(x) = (c(x)-1)/x$ and $c^{-1}(x) = 1 - xc(x)$, one can write

 $c^{n}(x) = p_{n-1}(x) + q_{n-1}(x)c(x)$

for $n \in \mathbb{Z}$. From $c^n(x) = c(x)c^{n-1}(x)$, one is led to $q_{n-1} = p_{n-2} + \frac{1}{x}q_{n-2}$ and $p_{n-1} = -\frac{1}{x}q_{n-2}$, or $q_{n-1} = (q_{n-2} - q_{n-3})/x$ with input $q_{-1} = 0$, $q_0 = 1$. So $q_{n-1}(x) = \mathcal{U}_{n-1}(x)$ and $p_{n-1}(x) = -\mathcal{U}_{n-2}(x)/x$. Equation (23) then follows from (21). \Box

Note 1: Because

$$S_n(y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} y^{n-2j},$$

the explicit form of these polynomials (2) is

$$p_{n-1}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^{j+1} {\binom{n-2-j}{j}} x^{-(n-1-j)}, \ p_{-1} = 1, \ p_0 = 0,$$

and

$$q_{n-1}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{j} \binom{n-1-j}{j} x^{-(n-1-j)}, \ q_{-1} = 0$$

For negative index one has, due to (20),

$$p_{-(n+1)}(x) = (\sqrt{x})^n S_n(1/\sqrt{x}) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} x^j$$

and

$$q_{-(n+1)}(x) = -(\sqrt{x})^{n+1} S_{n-1}(1/\sqrt{x}) = -x \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-1-j}{j} x^j.$$

In the Table, one can find the coefficient triangle for the polynomials $\{p_n(x)\}_{-1}^{12}$ with column *m* corresponding to $(\frac{1}{x})^m$, $m \ge 0$.

Note 2: An alternative proof of Proposition 1 can be given starting with (17) and (18) which show, together with $\lambda_{+}(x) - \lambda_{-}(x) = \sqrt{x^2 - 4}$, that

$$\lambda_{\pm}^{n}(x) = T_{n}(x/2) \pm \sqrt{(x/2)^{2} - 1} S_{n-1}(x), \qquad (24)$$

or, from $\pm \sqrt{(x/2)^2 - 1} = \lambda_{\pm}(x) - x/2$ and the S_n recurrence relation (15),

$$\lambda_{\pm}^{n}(x) = T_{n}(x/2) - \frac{1}{2} \left(S_{n}(x) + S_{n-2}(x) \right) + S_{n-1}(x)\lambda_{\pm}(x)$$
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$$= -S_{n-2}(x) + S_{n-1}(x)\lambda_{\pm}(x).$$
(26)

Now (23) follows from (16). This also proves that, in Proposition 1, one may replace c(x) by $c_+(x) = 1/(xc(x))$, from which one recovers the c^{-n} formula for $n \in \mathbb{N}$ in accordance with (19) and (20).

				,,,				,					
n\m	0	1	2	3	4	5	6	7	8	9	10	11	12
-1	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	-1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	-1	0	0	0	0	0	0	0	0	0	0
3	0	0	1	-1	0	0	0	0	0	0	0	0	0
4	0	0	0	2	-1	0	0	0	0	0	0	0	0
5	0	0	0	-1	3	-1	0	0	0	0	0	0	0
6	0	0	0	0	-3	4	-1	0	0	0	0	0	0
7	0	0	0	0	1	6	5	-1	0	0	0	0	0
8	0	0	0	0	0	4	-10	6	-1	0	0	0	0
9	0	0	0	0	0	-1	10	-15	7	-1	0	0	0
10	0	0	0	0	0	0	-5	20	-21	8	-1	0	0
11	0	0	0	0	0	0	1	-15	35	-28	9	-1	0
12	0	0	0	0	0	0	0	6	-35	56	-36	10	-1

TABLE. $p(n, m) = [1/x^m] p_{-}\{n\}(x)$ Coefficient Matrix n = -1, ..., 12, m = 0, ..., 12

Note 3: For the transfer matrix T(x), defined in (12), one can prove for $n \in \mathbb{N}$, in an analogous manner, that

$$\mathbf{T}^{n} = -\left(\frac{1}{\sqrt{x}}\right)^{n} S_{n-2}(1/\sqrt{x}) \mathbf{1} + \left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}(1/\sqrt{x}) \mathbf{T}(x),$$
(27)

by employing the Cayley-Hamilton theorem for the 2×2 matrix **T** with tr $\mathbf{T} = \frac{1}{x} = \det \mathbf{T}$, which states that **T** satisfies the characteristic equation $\mathbf{T}^2 - \frac{1}{x}\mathbf{T} + \frac{1}{x}\mathbf{I} = 0$.

Powers of a function which generates a sequence generate convolutions of this sequence. Therefore, Proposition 1 implies that convolutions of the Catalan sequence can be expressed in terms of Catalan numbers and binomial coefficients. Before giving this result, we shall present an explicit formula for the n^{th} convolution of a general sequence $\{C_l\}$ generated by $c(x) = \sum_{l=0}^{\infty} C_l x^l$. Usually, the convolution coefficients $C_l(n)$, defined by $c^n(x) = \sum_{l=0}^{\infty} C_l(n) x^l$, are written as

$$C_{l}(n) = \sum_{\sum_{j=1}^{n} i_{j}=l} C_{i_{1}} C_{i_{2}} \cdots C_{i_{n}}, \text{ with } i_{j} \in \mathbb{N}_{0}.$$
 (28)

An explicit formula with (l-1) nested sums is the content of the next lemma.

Lemma 1–General convolutions: For l = 2, 3, ...,

$$C_{l}(n) = C_{0}^{n-l} C_{1}^{l} \left(\prod_{k=2}^{l} \sum_{i_{k}=a_{k}}^{\lfloor b_{k} \rfloor} \right) \langle n, l, \{i_{j}\}_{2}^{l} \rangle \prod_{j=2}^{l} \left(\left(\frac{C_{j} C_{0}^{j-1}}{C_{1}^{j}} \right)^{i_{j}} \frac{1}{i_{j}!} \right),$$
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with

$$b_2 = l/2, \ b_k = \left(l - \sum_{j=2}^{k-1} ji_j\right)/k,$$
 (30)

$$a_k = 0$$
 for $k = 2, 3, ..., l-1; a_l = \max\left(0, \left\lceil \frac{l-n-\sum_{j=2}^{l-1}(j-1)i_j}{l-1} \right\rceil\right),$ (31)

$$\langle n, l, \{i_j\}_2^l \rangle = \frac{n!}{\left(n - l + \sum_{j=2}^l (j-1)i_j\right)! \left(l - \sum_{j=2}^l ji_j\right)!}.$$
 (32)

The first product in (29) is understood to be ordered such that the sums have indices $i_2, i_3, ..., i_l$ when written from the left to the right. In addition: $C_0(n) = C_0^n$ and $C_1(n) = nC_0^{n-1}C_1$.

Proof: $C_l(n)$ of (28) is rewritten first as

$$C_{l}(n) = \sum (n, l, \{i_{j}\}_{0}^{l}) C_{0}^{i_{0}} C_{1}^{i_{1}} \cdots C_{l}^{i_{l}}, \quad i_{j} \in \mathbb{N}_{0},$$
(33)

where the sum is restricted by

(i):
$$\sum_{j=0}^{l} ji_j = l$$
 and (ii): $\sum_{j=0}^{l} i_j = n.$ (34)

 $(n, l, \{i_j\}_2^l)$ is a combinatorial factor to be determined later on. (E.g., for n = 3, l = 5, one has five terms in the sum: $i_5 = 1, i_0 = 2$; $i_4 = 1, i_1 = 1, i_0 = 1$; $i_3 = 1, i_2 = 1, i_0 = 1$; $i_3 = 1, i_1 = 2$; $i_2 = 2, i_1 = 1$, with other indices vanishing, and the combinatorial factors are 3, 6, 6, 3, 3, respectively.) (*ii*) restricts the sum to terms with *n* factors, and (*i*) produces the correct weight *l*. These restrictions are solved by

(*i'*):
$$i_1 = l - \sum_{j=2}^{l} j i_j$$
 and (*ii'*): $i_0 = n - i_1 - \sum_{j=2}^{l} i_j = n - l + \sum_{j=2}^{l} (j-1) i_j$

From $i_1 \ge 0$, i.e., $l - \sum_{j=2}^l ji_j \ge 0$, one infers $i_2 \le \lfloor \frac{l}{2} \rfloor$; thus, $i_2 \in [0, \lfloor \frac{l}{2} \rfloor]$. For given i_2 in this range, $i_3 \le \lfloor \frac{l-2i_2}{2} \rfloor$, etc. In general,

$$0 \le i_k \le \left| \left(l - \sum_{j=2}^{k-1} j i_j \right) k \right|$$
 for $k = 2, 3, ..., l$

with the sum replaced by zero for k = 2. This accounts for the upper boundaries $\lfloor b_k \rfloor$ in (30). Now, because $i_0 \ge 0$, (*ii*') implies a lower bound for i_1 , the index of the last sum, viz

$$i_l \ge \left[\left(l - n - \sum_{j=2}^{l-1} (j-1)i_j \right) / (l-1) \right]$$

with the ceiling function $\lceil \cdot \rceil$. In any case $i_l \ge 0$; therefore, the lower boundary for the i_l -sum is a_l as given in (31). All restrictions have then been solved and the lower boundaries of the other sums are given by $a_k = 0$ for k = 2, ..., l-1. As to the combinatorial factor, it now depends only on $n, l, \{i_j\}_2^l$ and is written as $\langle n, l, \{i_j\}_2^l \rangle$. It counts the number of possibilities for the occurrence of the considered term of the sum which is given by

$$\binom{n}{i_0}\binom{n-i_0}{i_1}\cdots\binom{n-\sum_{j=0}^{l-1}i_j}{i_l} = n! / \left(\prod_{j=0}^{l}i_j!\right) \binom{n-\sum_{j=0}^{l}i_j}{i_j!}.$$

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Inserting i_0 and i_1 from (*ii'*) and (*i'*), respectively, remembering (*ii*), produces $\langle n, l, \{i_j\}_2^l \rangle$ as given in (32). Finally, $\sum \langle n, l, \{i_j\}_2^l \rangle C_0^{i_0} C_1^{i_1} \cdots C_l^{i_l}$ is transformed into (l-1) nested sums with boundaries a_k and $\lfloor b_k \rfloor$ after replacement of i_1 and i_0 . This completes the proof of (29) for the nontrivial $l \ge 2$ cases. \Box

Corollary 1-Catalan convolutions: For Catalan's sequence $\{C_n\}_0^\infty$, the n^{th} convolution sequence for $n \in \mathbb{N}$ is given by $C_0(n) = 1$, $C_1(n) = n$ and, for l = 2, 3, ..., by

$$C_l(n) = \left(\prod_{k=2}^l \sum_{i_k=a_k}^{\lfloor b_k \rfloor}\right) \langle n, l, \{i_j\}_2^l \rangle \prod_{j=2}^l \left(\frac{C_j^{i_j}}{i_j!}\right),$$
(35)

with (30), (31), and (32).

Proof: This is Lemma 1 with $C_0 = 1 = C_1$. \Box

Example 1: $C_4(3) = 3C_4 + 5C_3 + 3C_2^2 + 3C_2 = 90.$

Corollary 2: With the Catalan generating function c(x) and the definition, one has, for $n \in \mathbb{N}$, $c^{-n}(x) =: \sum_{l=0}^{\infty} C_l(-n)x^l$, for l = 2, 3, ...,

$$C_{l}(-n) = (-1)^{l} \left(\prod_{k=2}^{l} \sum_{i_{k}=a_{k}}^{\lfloor b_{k} \rfloor} \frac{(-1)^{(k-1)i_{k}}}{i_{k}!} \right) \langle n, l, \{i_{j}\}_{2}^{l} \rangle \prod_{j=2}^{l-1} C_{j}^{i_{j+1}},$$
(36)

with (30), (31), (32), and Catalan numbers C_k . In addition, $C_0(-n) = 1$, $C_1(-n) = -n$.

Proof: Lemma 1 is used for powers of c(x) replaced by those of $c^{-1}(x) = 1 - xc(x)$, with the Catalan generating function c(x). Hence, $c^{-1}(x) = \sum_{k=0}^{\infty} C_k(-1)x^k$ with

$$C_k(-1) = \begin{cases} 1 & \text{for } k = 0, \\ -C_{k-1} & \text{for } k = 1, 2, \dots \end{cases}$$
 Then, in Lemma 1, C_k is replaced by $C_k(-1)$. \Box

Example 2: $C_4(-3) = -3C_3 + 6C_2 - 3 + 3 = -3$.

Convolutions of Catalan's sequence have been encountered in various contexts, for example, in the enumeration of nonintersecting path pairs on a square lattice (see [12], [18], [5]), and in the problem of inverting triangular matrices with Pascal triangle entries (see [6] and earlier works cited there; they also appear in [15], p. 148).

Note 4: Shapiro's Catalan triangle has entries

$$B_{n,k} = \frac{k}{n} \binom{2n}{n-k}$$
 for $n \ge k \ge 1$, and $B_{n,k} = [x^n](x^k \hat{c}^k(x))$,

with $[x^n]f(x)$ denoting the coefficient of x^n in the expansion of f(x) around x = 0. In this case, $\hat{c}(x) = (c(x)-1)/x = c^2(x)$. (See [12], Propositions (2.1) and (3.3), with $i_j \in \mathbb{N}$, not \mathbb{N}_0 .) In [18] this triangle of numbers from [12] reappears as b(n, k), and it is shown there that $B_{n,k} \equiv b(n, k) = [x^n](xc^2(x))^k$, in accordance with the identity $\hat{c}(x) = c^2(x)$. Therefore, only even powers of c(x)appear in Shapiro's Catalan triangle. In [5], $C_l(n)$ appears as special case ${}_2d_{2-n,l+1}$. In [6], all powers of c(x) show up as convolutions for the special case of the S_1 sequence there. The entries of the S_1 -array ([6], p. 397) are $[x^n]c^{k+1}(x)$ for $n, k \in \mathbb{N}_0$.

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The anonymous referee of this paper noticed that the inverse of the lower triangular matrix $S_{n,k} = [x^k]S_n(x)$, for $n, k \in \mathbb{N}_0$, with Chebyshev's $S_n(x) = U_n(x/2)$ polynomials is the lower triangular convolution matrix obtained from its first (k = 0) column sequence generated by $c(x^2)$ (Catalan numbers alternating with zeros). This follows from the fact that the S-matrix is also a lower triangular convolution matrix with generating function $1/(1+x^2)$ of its first column. See [13] for such type of matrices **M** and the relation between the generating functions of the first columns of **M** and \mathbf{M}^{-1} . The head of this Catalan triangle can be viewed under number A053121 in the on-line database [14]. See also [6] for inverses of Pascal-type arrays.

Lemma 2-Explicit form of Catalan convolutions [12], [18], [6], [4], [11], and [5]:

For $n \in \mathbf{R}$, $l \in \mathbf{N}_0$:

$$C_{l}(n) = \frac{n}{l} \binom{2l+n-1}{l-1} = \frac{n}{n+2l} \binom{n+2l}{l} = \frac{n}{l+n} \binom{2l+n-1}{l}.$$
(37)

Proof: Three equivalent expressions have been given for convenience. See [4], page 201, equation (5.60), with $\mathcal{B}_2(z) = c(z), t \to 2, k \to l, r \to n$. The proof of (5.60) appears as (7.69) on page 349 of [4], with $m = 2, n = l \in \mathbb{R}$.

The same formula occurs as Exercise 213 in Vol. 1 of [11] for $\beta = 2$ as a special instance of Exercises 211 and 212. Put $\alpha = n$ and n = l in the solution of Exercise 213 on page 301.

In order to prove this lemma from [12] or [18], one can use

$$C_l(n) = \sum_{j=0}^{\min(l,n)} \binom{n}{j} \hat{C}_l(j)$$

obtained from $c(x) = 1 + \hat{c}(x)$ with

$$\hat{c}^n(x) =: \sum_{k=n}^{\infty} \hat{C}_k(n) x^{k-n}.$$

The result in [12] and [18] is, with this notation,

$$\hat{C}_{l}(j) = B_{l,j} = b(l,j) = \frac{j}{l} \binom{2l}{l-j}.$$

Inserting this in the given sum, making use of the identity $j\binom{n}{j} = n\binom{n-1}{j-1}$ and the Vandermonde convolution identity, leads to Lemma 2 at least for positive integer *n*, but one can continue this formula to real (or complex) *n*.

In [6], one finds this result as equation (3.1), page 402, for i = 1: $s_1(l, n) = C_l(n)$.

In [5], $_{2}d_{2-n,l+1} = C_{l}(n)$, with the result given in Theorem 2.3, equation (2.6), page 71. \Box

Note 5: As a side remark we mention that, from (37), $E_l(x) := l! C_l(x)$ (with real n = x) is a polynomial of degree l, viz $\prod_{j=0}^{l-1} (x+l+1+j)$. These polynomials, which are not the subject of this work, are known (see [8] and references given there) as exponential convolution polynomials satisfying $E_l(x+y) = \sum_{k=0}^{l} {l \choose k} E_k(x) E_{l-k}(y)$.

We now compute the coefficients $C_l(n) = [x^l]c^n(x)$ (see Note 4 for this notation) from our formula given in Proposition 1. This can be done for $n \in \mathbb{Z}$.

First, consider $n \in \mathbb{N}_0$. For n = 0 and n = 1, there is nothing new due to the inputs $S_{-2} = -1$, $S_{-1} = 0$, and $S_0 = 1$. $C_l(n) = 0$ for negative integer *l*. Therefore, terms proportional to $1/x^l$ with

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 $l \in \mathbb{N}$ have to cancel in (23), or in (1). For n = 2, 3, ..., terms of the type $1/x^{n-j}$ occur for $j \in \{1, 2, ..., \lfloor \frac{n}{2} \rfloor\}$. The coefficient of $1/x^{n-j}$ in $p_{n-1}(x)$ is $(-1)^j \binom{n-1-j}{j-1}$ (see Note 1 for the explicit form of p_{n-1}). For the $1/x^{n-j}$ coefficient in $q_{n-1}(x)c(x)$, one finds the convolution

$$\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-(j-l)}{j-l-1} C_l.$$

Compensation of both coefficients leads to identity (P1) given in (4) after (j-1) has been traded for p. Thus, after a shift $n \rightarrow n+2$,

Proposition 2-Identity (P1): For $n \in \mathbb{N}_0$ and $p = 0, 1, ..., \lfloor \frac{n}{2} \rfloor$, identity (P1), given in equation (4), holds.

Example 3: n = 2(k-1), p = k-1, and n = 2k-1, p = k-1 for $k \in \mathbb{N}$;

$$\sum_{l=0}^{k-1} (-1)^l \binom{k+l}{2l+1} C_l = 1, \qquad \sum_{l=0}^{k-1} (-1)^l \binom{k+l+1}{2(l+1)} C_l = k;$$

e.g., k = 3: $3C_0 - 4C_1 + 1C_2 = 1$, $6C_0 - 5C_1 + 1C_2 = 3$.

For n = 2, 3, ..., terms in (1), or in (23), proportional to x^k with $k \in \mathbb{N}_0$ arise only from $q_{n-1}(x)c(x)$, and they are given by the convolution (cf. Note 1),

$$\sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{l} \binom{n-1-l}{l} C_{k+n-1-l}.$$

For n = 1, this is C_k . The left-hand side of (1) contributes $C_k(n)$, and $C_k(1) = C_k$. Therefore,

Proposition 3-Identity (P3): For $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, identity (P2), given in equation (5) with (3) holds.

Example 4: $k = 0, (n-1) \rightarrow n$:

$$\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{l+1} \binom{n-l}{l} C_{n-l} = C_n - 1;$$

e.g., n = 3: $2C_2 = C_3 - 1$, n = 4: $3C_3 - 1C_2 = C_4 - 1$.

Now consider negative powers in (1), i.e., $c^{-n}(x)$, $n \in \mathbb{N}$. No negative powers of x appear (cf. Note 1 for the explicit form of $p_{-(n+1)}(x)$ and $q_{-(n+1)}(x)$). The coefficient of x^k , $k \in \mathbb{N}_0$, of the right-hand side of (1) is

$$(-1)^{k}\binom{n-k}{k} - \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{l}\binom{n-1-l}{l} C_{k-1-l},$$

where the first term, arising from $p_{-(n+1)}(x)$, contributes only for $k \in \{0, 1, ..., \lfloor \frac{n}{2} \rfloor\}$. In the summation, one also needs $l \le k-1$ because no Catalan numbers with negative index occur in (1). The left-hand side of (1) has $[x^k]c^{-n}(x) = C_k(-n)$. From the last equation in (37), one finds

$$C_{k}(-n) = \frac{n}{n-k} \binom{2k-n-1}{k} = (-1)^{k} \frac{n}{n-k} \binom{n-k}{k}.$$

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In the last equation, the upper index in the binomial has been negated (cf. [4], (5.14)). Two sets of identities follow, depending on the range of k.

Proposition 4–Identity (P3): For $n \in \mathbb{N}$, $k \in \{0, 1, ..., \lfloor \frac{n}{2} \rfloor\}$, identity (P3), given in equation (6), holds.

Example 5: $k = 3, n \ge 6$: $C_2 - (n-2)C_1 + \binom{n-3}{2}C_0 = \binom{n-4}{2}$.

Proposition 5-Identity (P4): For $n \in \mathbb{N}$, $k \in \mathbb{N}$, with $k \ge \lfloor \frac{n}{2} \rfloor + 1$, identity (P4), given in equation (7), holds.

In (P4), only the $q_{-(n+1)}(x)c(x)$ part of (1) contributed, and we used the first expression for $C_k(-n)$ in (37). In (P3), where $p_{-(n+1)}(x)$ also contributed, we used the negated binomial coefficient for $C_l(-n)$ and absorption in the resulting one.

Note that (37) implies $C_k(-n) = -C_{k-n}(n)$ for $k, n \in \mathbb{N}$, $k \ge n$, and $C_k(0) = \delta_{k,0}$.

Example 6: $n = 5, k \ge 3$:

$$C_{k-1} - 3C_{k-2} + C_{k-3} = \frac{5}{k} \binom{2k-6}{k-1};$$

e.g., k = 7: $C_6 - 3C_5 + C_4 = 20$.

If one uses the binomial formula for $c^{-n}(x) = (1 - xc(x))^n$ and $c^n(x) = \sum_{k=0}^{\infty} C_k(n) x^k$, one arrives at equation (8).

3. SOME FAMILIES OF INTEGER SEQUENCES

In this section we present some sequences of positive integers which are defined with the help of the U_n polynomials (10).

$$u_n(m) := \mathcal{U}_n(1/m) = (\sqrt{m})^n S_n(\sqrt{m}).$$
 (38)

The last equation is due to (21). It will be shown that $u_n(m)$ is a nonnegative integer for each m = 4, 5, ... and n = -1, 0, ... Also negative integers $-m, m \in \mathbb{N}$ are of interest. In this case, we add a sign factor:

$$v_n(m) := (-1)^n \mathcal{U}_n(-1/m) = (-i\sqrt{m})^n S_n(i\sqrt{m}).$$
(39)

From the S_n recursion relation (15), one infers those for the $u_n(m)$ and $v_n(m)$ sequences:

$$u_n(m) = m(u_{n-1}(m) - u_{n-2}(m)), \quad u_{-1}(m) \equiv 0, \quad u_0(m) \equiv 1;$$
(40)

$$v_n(m) = m(v_{n-1}(m) + v_{n-2}(m)), \quad v_{-1}(m) \equiv 0, \quad v_0(m) \equiv 1.$$
 (41)

This shows that $v_n(m)$ constitutes a nonnegative integer sequence for positive integer m. It describes certain generalized Fibonacci sequences (see, e.g., [7] with $v_n(m) = W_{n+1}(0, 1; m, m)$). For example, $v_n(m)$ counts the length of the binary word W(m, n) obtained at step n from the substitution rule $1 \rightarrow 1^m 0$, $0 \rightarrow 1^m$, starting at step n = 0 with 0. The number of 1's, resp. 0's, in W(m; n) is $mv_{n-1}(m)$, resp. $mv_{n-2}(m)$. E.g., $W(2; 3) = (110)^2 1^2 (110)^2 1^2$ and $v_3(2) = 16$, $2v_2(2) = 12$, and $2v_1(2) = 4$. For m = 1, this substitution rule produces the well-known Fibonacci-tree. Of course, one can define in a similar manner generalized Lucas sequences using the polynomials $\{v_n\}$ given in (11). Each $u_n(m)$ sequence (which is identified with $W_{n+1}(0, 1; m, -m)$ of [7]) turns

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out to be composed of two simpler sequences, viz $u_{2k}(m) = m^k \alpha_k(m)$ and $u_{2k-1}(m) = m^k \beta_k(m)$, $k \in \mathbb{N}_0$. These new sequences, which are due to (38), given by $\alpha_k = S_{2k}(\sqrt{m})$ and $\beta_k(m) = S_{2k-1}(\sqrt{m})/\sqrt{m}$, satisfy therefore the following relations:

$$\beta_{k+1}(m) = (m-2)\beta_k(m) - \beta_{k-1}(m), \ \beta_0(m) \equiv 0, \ \beta_1(m) \equiv 1,$$
(42)

and

$$\alpha_{k-1}(m) = \beta_k(m) + \beta_{k-1}(m). \tag{43}$$

From (42) it is now clear that $\beta_n(m)$ is a nonnegative integer sequence for m = 4, 5, ... (In [7], $\beta_n(m) = W_n(0, 1; m-2, -1)$.) This property is then inherited by the $\alpha_n(m)$ sequences due to (43), and then by the composed sequence $u_n(m)$.

The ordinary generating functions are:

$$g_{\beta}(m;x) := \sum_{n=0}^{\infty} \beta_n(m) x^n = \frac{x}{x^2 - (m-2)x + 1}, \quad g_{\alpha}(m;x) := \sum_{n=0}^{\infty} \alpha_n(m) x^n = \frac{1+x}{x^2 - (m-2)x + 1}; \quad (44)$$

$$g_u(m;x) := \sum_{n=0}^{\infty} u_n(m) x^n = \frac{1}{1 - mx + mx^2}, \qquad g_v(m;x) := \sum_{n=0}^{\infty} v_n(m) x^n = \frac{1}{1 - mx - mx^2}.$$
 (45)

Note 6: The $\{\beta_n(m)\}\$ sequences for m = 4, 5, 6, 7, 8, 10 appear in the book [14]. The case m = 4 produces the sequence of nonnegative integers; m = 5 are the even-indexed Fibonacci numbers. The m = 9 sequence appears in Sloane's "On-Line-Encyclopedia" [14] as A004187. The $\{\alpha_n(m)\}\$ sequences for m = 4, 5, 6, 8 also appear in the book [14]. The case m = 4 yields the positive odd integer sequence; m = 5 is the odd-indexed Lucas number sequence. The m = 7 sequence appears in the database [14] as A030221. The composed sequences $\{u_n(m)\}\$ do not appear in the book [14], but some of them are found in the database [14]. m = 4 is the sequence $(n + 1)2^n$, A001787, and m = 5, 6, 7 appear as A030191, A030192, and A030140, respectively. As mentioned above, $\{v_{n-1}(1)\}\$ is the Fibonacci sequence. The instances m = 2 and 3 appear as A002605 and A030195, respectively, in the database [14].

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