# ON POLYNOMIALS RELATED TO POWERS OF THE GENERATING FUNCTION OF CATALAN'S NUMBERS 

Wolfdieter Lang<br>Institut für Theoretische Physik, Universität Karlsruhe<br>Kaiserstrasse 12, D-76128 Karlsruhe, Germany<br>E-mail: wolfdieter.lang@physik.uni-karlsruhe.de<br>(Submitted May 1998-Final Revision May 2000)

## 1. INTRODUCTION AND SUMMARY

Catalan's sequence of numbers $\left\{C_{n}\right\}_{0}^{\infty}=\{1,1,2,5,14,42, \ldots\}$ (nr. 1459 and $A 000108$ of [14]) emerges in the solution of many combinatorial problems (see [2], [4], [5], and [16] for further references). The moments $\mu_{2 k}$ of the normalized weight function of Chebyshev's polynomials of the second kind are given by $C_{k} / 2^{k}$ (see, e.g., [3], Lemma 4.3, p. 160 for $l=0$, and [17], p. II3). This sequence also shows up in the asymptotic moments of zeros of scaled Laguerre and Hermite polynomials (see [9], eqs. (3.34) and (3.35)). The generating function $c(x)=\sum_{n=0}^{\infty} C_{n} x^{n}$ is the solution of the quadratic equation $x c^{2}(x)-c(x)+1=0$ with $c(0)=1$. Therefore, every positive integer power of $c(x)$ can be written as

$$
\begin{equation*}
c^{n}(x)=p_{n-1}(x) 1+q_{n-1}(x) c(x), \tag{1}
\end{equation*}
$$

with certain polynomials $p_{n-1}$ and $q_{n-1}$, both of degree $(n-1)$, in $1 / x$. In Section 2, they are shown to be related to Chebyshev polynomials of the second kind:

$$
\begin{equation*}
p_{n-1}(x)=-\left(\frac{1}{\sqrt{x}}\right)^{n} S_{n-2}\left(\frac{1}{\sqrt{x}}\right), q_{n-1}(x)=\left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}\left(\frac{1}{\sqrt{x}}\right)=-x p_{n}(x), \tag{2}
\end{equation*}
$$

with $S_{n}(y)=U_{n}(y / 2)$. Therefore, it is possible to extend the range of the power $n$ to negative integers (or to real or complex numbers). Tables for the $U_{n}(x)$ polynomials can be found, e.g., in [1]. Because powers of a generating function correspond to convolutions of the generated number sequence, the given decomposition of $c^{n}(x)$ will determine convolutions of the Catalan sequence. In passing, an explicit expression for general convolutions in the form of nested sums will also be given. Contact with the works of [6], [12], [18], and [5] will be made.

Together with the known (e.g., [4], [11]) result (valid for real $n$ ),

$$
\begin{equation*}
c^{n}(x)=\sum_{k=0}^{\infty} C_{k}(n) x^{k}, \text { with } C_{k}(n)=\frac{n}{n+2 k}\binom{n+2 k}{k}=\frac{n}{k+n}\binom{n-1+2 k}{k}, \tag{3}
\end{equation*}
$$

one finds, from the alternative expression (1) for positive $n$, two sets of identities:

$$
\begin{equation*}
\sum_{l=0}^{p}(-1)^{l}\binom{n+1-p+l}{p-l} C_{l}=\binom{n-p}{p} \tag{P1}
\end{equation*}
$$

for $n \in \mathbf{N}_{0}, p \in\left\{0,1,2, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$, and

$$
\begin{equation*}
\sum_{l=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{l}\binom{n-1-l}{l} C_{k+n-1-l}=C_{k}(n) \tag{P2}
\end{equation*}
$$

for $n \in \mathbf{N}, k \in \mathbf{N}_{0}$.

For negative powers in (1), two other sets of identities result:
(P3) $\quad \sum_{l=0}^{\min \left[\begin{array}{l}n-1 \\ 2\end{array}, k-1\right)}(-1)^{l}\binom{n-1-l}{l} C_{k-1-l}=(-1)^{k+1}\binom{n-k-1}{k-1}$
for $n \in \mathbf{N}, k \in\left\{0,1,2, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$ (for $k=0$, both sides are by definition zero), and

$$
\begin{equation*}
\text { (P4) } \quad \sum_{l=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{l}\binom{n-1-l}{l} C_{k-1-l}=-C_{k}(-n)=\frac{n}{k}\binom{2 k-n-1}{k-1} \tag{7}
\end{equation*}
$$

for $n \in \mathbf{N}, k \in \mathbf{N}$ with $k \geq\left\lfloor\frac{n}{2}\right\rfloor+1$. These identities can be continued for appropriate values of real $n$.

Another expression for the coefficients of negative powers of $c(x)$ is

$$
\begin{equation*}
C_{k}(-n)=\sum_{l=1}^{\min (n, k)}(-1)^{l}\binom{n}{l} C_{k-l}(n) \tag{8}
\end{equation*}
$$

for $n, k \in \mathbf{N}$, and $C_{0}(-n)=1, C_{n}(0)=\delta_{n, 0}$. Also, from (3), $C_{k}(-n)=-C_{k-n}(n)$ for $n, k \in \mathbf{N}$ with $k \geq n$.

The remainder of this paper provides proofs for the above given statements. Section 2 deals with integer (and real) powers of the generating function $c(x)$. Convolutions of general sequences are expressed there in terms of nested sums. In Section 3 some families of integer sequences related to the polynomials $q_{n}(x)$ (2) evaluated for $x=1 / m$ for $m=4,5, \ldots$ and $(-1)^{n} q_{n}(x)$ evaluated at $x=-1 / m, m \in \mathbf{N}$, are considered.

## 2. POWERS

The equation $x c^{2}(x)-c(x)+1=0$ whose solution defines the generating function of Catalan's numbers if $c(0)=1$ can be considered as a characteristic equation for the recursion relation

$$
\begin{equation*}
x r_{n+1}-r_{n}+r_{n-1}=0, n=0,1, \ldots, \tag{9}
\end{equation*}
$$

with arbitrary inputs $r_{-1}(x)$ and $r_{0}(x)$. A basis of two linearly independent solutions is given by the Lucas-type polynomials $\left\{u_{n}\right\}$ and $\left\{\nu_{n}\right\}$ with standard inputs $u_{-1}=0, u_{0}=1,\left(u_{-2}=-x\right)$, and $v_{-1}=1, v_{0}=2,\left(v_{1}=1 / x\right)$, in the Binet form

$$
\begin{gather*}
u_{n-1}(x)=\frac{c_{+}^{n}(x)-c_{-}^{n}(x)}{c_{+}(x)-c_{-}(x)}  \tag{10}\\
v_{n}(x)=c_{+}^{n}(x)+c_{-}^{n}(x)=\frac{1}{x}\left(u_{n-1}(x)-2 u_{n-2}(x)\right), \tag{11}
\end{gather*}
$$

with the two solutions of the characteristic equation, viz $c_{ \pm}(x):=(1 \pm \sqrt{1-4 x}) /(2 x) . c(x):=c_{-}(x)$ satisfies $c(0)=1$, and $c_{+}(x)=1 /(x c(x))$, as well as $c_{+}(x)+c(x)=1 / x$. From the recurrence (9), it is clear that, for positive $n \neq 0, u_{n}$ is a polynomial in $1 / x$ of degree $n-1$. If $c_{+}(x)-c_{-}(x)=0$, i.e., $x=1 / 4$, equation (10) is replaced by $u_{n}(1 / 4)=2^{n}(n+1)$. The second equation in (11) holds because both sides of the equation satisfy recurrence (9) and the inputs for $v_{0}$ and $v_{1}$ match. One may associate with the recurrence relation (9) a transfer matrix

$$
\mathbf{T}(x)=\left(\begin{array}{cc}
1 / x & -1 / x  \tag{12}\\
1 & 0
\end{array}\right), \operatorname{det} \mathbf{T}(x)=1 / x
$$

With this matrix, one can rewrite (9) as

$$
\begin{equation*}
\binom{r_{n}}{r_{n-1}}=\mathbf{T}(x)\binom{r_{n-1}}{r_{n-2}}=\mathbf{T}^{n}(x)\binom{r_{0}(x)}{r_{-1}(x)} . \tag{13}
\end{equation*}
$$

Because $\mathbf{T}^{n}=\mathbf{T} \mathbf{T}^{n-1}$ with input $\mathbf{T}^{1}=\mathbf{T}(x)$ given by (12), one finds from the recurrence relation (9) with $r_{n}=u_{n}$ that

$$
\mathbf{T}^{n}(x)=\left(\begin{array}{cc}
u_{n}(x) & -\frac{1}{x} u_{n-1}(x)  \tag{14}\\
u_{n-1}(x) & -\frac{1}{x} u_{n-2}(x)
\end{array}\right) .
$$

Note that, for $x=1$, one has $c_{ \pm}(1)=(1 \pm i \sqrt{3}) / 2$, which are $6^{\text {th }}$ roots of unity, and the related period 6 sequences are $\left\{u_{n}(1)\right\}_{-1}^{\infty}=\{\overline{0,1,1,0,-1,-1}\}$, as well as $\left\{v_{n}(1)\right\}_{0}^{\infty}=\{\overline{2,1,-1,-2,-1,1}\}$. This follows from equations (10) and (11). It is convenient to map the recursion relation (9) to the familiar one for Chebyshev's $S_{n}(x)=U_{n}(x / 2)$ polynomials of the second kind, viz

$$
\begin{equation*}
S_{n}(x)=x S_{n-1}(x)-S_{n-2}(x), S_{-1}=0, S_{0}=1 \tag{15}
\end{equation*}
$$

with characteristic equation $\lambda^{2}-x \lambda+1=0$ and solutions $\lambda_{ \pm}(x)=\frac{x}{2}\left(1 \pm \sqrt{1-(2 / x)^{2}}\right)$, satisfying $\lambda_{+}(x) \lambda_{-}(x)=1$ and $\lambda_{+}(x)+\lambda_{-}(x)=x$. The relation to $c_{ \pm}(x)$ is

$$
\begin{equation*}
\sqrt{x} c_{ \pm}(x)=\lambda_{ \pm}(1 / \sqrt{x}) \tag{16}
\end{equation*}
$$

The Binet form of the corresponding two independent polynomial systems is

$$
\begin{gather*}
S_{n-1}(x)=\frac{\lambda_{+}^{n}(x)-\lambda_{-}^{n}(x)}{\lambda_{+}(x)-\lambda_{-}(x)}  \tag{17}\\
2 T_{n}(x / 2)=\lambda_{+}^{n}(x)+\lambda_{-}^{n}(x) \tag{18}
\end{gather*}
$$

and $T_{n}(x / 2)=\left(S_{n}(x)-S_{n-2}(x)\right) / 2$ are Chebyshev polynomials of the first kind. Tables of Chebyshev polynomials can be found in [1]. The coefficient triangles of the $S_{n}(x), U_{n}(x)$, and $T_{n}(x)$ polynomials can also be viewed under the numbers A049310, A053117, and A053120, respectively, in the on-line database [14].

The extension to negative indices runs as follows:

$$
\begin{gather*}
u_{-n}(x)=-x^{n-1} u_{n-2}(x)  \tag{19}\\
S_{-(n+2)}(x)=-S_{n}(x) \tag{20}
\end{gather*}
$$

This follows from (10) and (17). Note that from (9), $u_{n}$ is for positive $n$ a monic polynomial in $1 / x$ of degree $n$, and for negative $n$ in general, a nonmonic polynomial in $x$ of degree $\left\lfloor-\frac{n}{2}\right\rfloor$. It is possible to extend the range of $n$ to complex numbers using the Binet forms.

A connection between both systems of polynomials is made, using (10), (16), and (17), by

$$
\begin{equation*}
u_{n}(x)=\left(\frac{1}{\sqrt{x}}\right)^{n} S_{n}(1 / \sqrt{x}) \tag{21}
\end{equation*}
$$

This holds for $n \in \mathbf{Z}$ in accordance with (19) and (20).

After these preliminaries, we are ready to state the following proposition.
Proposition 1: The $n^{\text {th }}$ power of $c(x)$, the generating function of Catalan numbers can, for $n \in \mathbb{Z}$, be written as

$$
\begin{align*}
c^{n}(x) & =-\frac{1}{x} u_{n-2}(x)+u_{n-1}(x) c(x)  \tag{22}\\
& =-\left(\frac{1}{\sqrt{x}}\right)^{n} S_{n-2}(1 / \sqrt{x})+\left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}(1 / \sqrt{x}) c(x) \tag{23}
\end{align*}
$$

Proof: Due to $c^{2}(x)=(c(x)-1) / x$ and $c^{-1}(x)=1-x c(x)$, one can write

$$
c^{n}(x)=p_{n-1}(x)+q_{n-1}(x) c(x)
$$

for $n \in \mathbb{Z}$. From $c^{n}(x)=c(x) c^{n-1}(x)$, one is led to $q_{n-1}=p_{n-2}+\frac{1}{x} q_{n-2}$ and $p_{n-1}=-\frac{1}{x} q_{n-2}$, or $q_{n-1}=\left(q_{n-2}-q_{n-3}\right) / x$ with input $q_{-1}=0, q_{0}=1$. So $q_{n-1}(x)=u_{n-1}(x)$ and $p_{n-1}(x)=-u_{n-2}(x) / x$. Equation (23) then follows from (21).

Note 1: Because

$$
S_{n}(y)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^{j}\binom{n-j}{j} y^{n-2 j},
$$

the explicit form of these polynomials (2) is

$$
p_{n-1}(x)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^{j+1}\binom{n-2-j}{j} x^{-(n-1-j)}, p_{-1}=1, p_{0}=0
$$

and

$$
q_{n-1}(x)=\sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{j}\binom{n-1-j}{j} x^{-(n-1-j)}, q_{-1}=0 .
$$

For negative index one has, due to (20),

$$
p_{-(n+1)}(x)=(\sqrt{x})^{n} S_{n}(1 / \sqrt{x})=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^{j}\binom{n-j}{j} x^{j}
$$

and

$$
q_{-(n+1)}(x)=-(\sqrt{x})^{n+1} S_{n-1}(1 / \sqrt{x})=-x \sum_{j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{j}\binom{n-1-j}{j} x^{j} .
$$

In the Table, one can find the coefficient triangle for the polynomials $\left\{p_{n}(x)\right\}_{-1}^{12}$ with column $m$ corresponding to $\left(\frac{1}{x}\right)^{m}, m \geq 0$.
Note 2: An alternative proof of Proposition 1 can be given starting with (17) and (18) which show, together with $\lambda_{+}(x)-\lambda_{-}(x)=\sqrt{x^{2}-4}$, that

$$
\begin{equation*}
\lambda_{ \pm}^{n}(x)=T_{n}(x / 2) \pm \sqrt{(x / 2)^{2}-1} S_{n-1}(x), \tag{24}
\end{equation*}
$$

or, from $\pm \sqrt{(x / 2)^{2}-1}=\lambda_{ \pm}(x)-x / 2$ and the $S_{n}$ recurrence relation (15),

$$
\begin{equation*}
\lambda_{ \pm}^{n}(x)=T_{n}(x / 2)-\frac{1}{2}\left(S_{n}(x)+S_{n-2}(x)\right)+S_{n-1}(x) \lambda_{ \pm}(x) \tag{25}
\end{equation*}
$$

$$
\begin{equation*}
=-S_{n-2}(x)+S_{n-1}(x) \lambda_{ \pm}(x) \tag{26}
\end{equation*}
$$

Now (23) follows from (16). This also proves that, in Proposition 1, one may replace $c(x)$ by $c_{+}(x)=1 /(x c(x))$, from which one recovers the $c^{-n}$ formula for $n \in \mathbf{N}$ in accordance with (19) and (20).

$$
\begin{aligned}
& \text { TABLE. } p(n, m)=\left[1 / x^{m}\right] p_{-}\{n\}(x) \text { Coefficient Matrix } \\
& n=-1, \ldots, 12, m=0, \ldots, 12
\end{aligned}
$$

Note 3: For the transfer matrix $\mathbf{T}(x)$, defined in (12), one can prove for $n \in \mathbf{N}$, in an analogous manner, that

$$
\begin{equation*}
\mathbf{T}^{n}=-\left(\frac{1}{\sqrt{x}}\right)^{n} S_{n-2}(1 / \sqrt{x}) \mathbf{1}+\left(\frac{1}{\sqrt{x}}\right)^{n-1} S_{n-1}(1 / \sqrt{x}) \mathbf{T}(x) \tag{27}
\end{equation*}
$$

by employing the Cayley-Hamilton theorem for the $2 \times 2$ matrix $\mathbf{T}$ with $\operatorname{tr} \mathbf{T}=\frac{1}{x}=\operatorname{det} \mathbf{T}$, which states that $\mathbf{T}$ satisfies the characteristic equation $\mathbf{T}^{2}-\frac{1}{x} \mathbf{T}+\frac{1}{x} \mathbf{1}=0$.

Powers of a function which generates a sequence generate convolutions of this sequence. Therefore, Proposition 1 implies that convolutions of the Catalan sequence can be expressed in terms of Catalan numbers and binomial coefficients. Before giving this result, we shall present an explicit formula for the $n^{\text {th }}$ convolution of a general sequence $\left\{C_{l}\right\}$ generated by $c(x)=\sum_{l=0}^{\infty} C_{l} x^{l}$. Usually, the convolution coefficients $C_{l}(n)$, defined by $c^{n}(x)=\sum_{l=0}^{\infty} C_{l}(n) x^{l}$, are written as

$$
\begin{equation*}
C_{l}(n)=\sum_{\sum_{j=1}^{n}=i_{j}=l} C_{i_{1}} C_{i_{2}} \cdots C_{i_{n}}, \text { with } i_{j} \in \mathbf{N}_{0} . \tag{28}
\end{equation*}
$$

An explicit formula with $(l-1)$ nested sums is the content of the next lemma.
Lemma 1-General convolutions: For $l=2,3, \ldots$,

$$
\begin{equation*}
C_{l}(n)=C_{0}^{n-l} C_{1}^{l}\left(\prod_{k=2}^{l} \sum_{i_{k}=a_{k}}^{\left\lfloor b_{k}\right\rfloor}\right)\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle \prod_{j=2}^{l}\left(\left(\frac{C_{j} C_{0}^{j-1}}{C_{1}^{j}}\right)^{i_{j}} \frac{1}{i_{j}!}\right), \tag{29}
\end{equation*}
$$

with

$$
\begin{gather*}
b_{2}=l / 2, b_{k}=\left(l-\sum_{j=2}^{k-1} j i_{j}\right) / k  \tag{30}\\
a_{k}=0 \text { for } k=2,3, \ldots, l-1 ; a_{l}=\max \left(0,\left\lceil\frac{l-n-\sum_{j=2}^{l-1}(j-1) i_{j}}{l-1}\right\rceil\right),  \tag{31}\\
\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle=\frac{n!}{\left(n-l+\sum_{j=2}^{l}(j-1) i_{j}\right)!\left(l-\sum_{j=2}^{l} j i_{j}\right)!} . \tag{32}
\end{gather*}
$$

The first product in (29) is understood to be ordered such that the sums have indices $i_{2}, i_{3}, \ldots, i_{l}$ when written from the left to the right. In addition: $C_{0}(n)=C_{0}^{n}$ and $C_{1}(n)=n C_{0}^{n-1} C_{1}$.

Proof: $C_{l}(n)$ of (28) is rewritten first as

$$
\begin{equation*}
C_{l}(n)=\Sigma\left(n, l,\left\{i_{j}\right\}_{0}^{l}\right) C_{0}^{i_{0}} C_{1}^{i_{1}} \cdots C_{l}^{i_{l}}, \quad i_{j} \in \mathbf{N}_{0} \tag{33}
\end{equation*}
$$

where the sum is restricted by

$$
\begin{equation*}
\text { (i): } \sum_{j=0}^{l} j i_{j}=l \text { and (ii): } \sum_{j=0}^{l} i_{j}=n . \tag{34}
\end{equation*}
$$

$\left(n, l,\left\{i_{j}\right\}_{2}^{l}\right)$ is a combinatorial factor to be determined later on. (E.g., for $n=3, l=5$, one has five terms in the sum: $i_{5}=1, i_{0}=2 ; i_{4}=1, i_{1}=1, i_{0}=1 ; i_{3}=1, i_{2}=1, i_{0}=1 ; i_{3}=1, i_{1}=2 ; i_{2}=2, i_{1}=1$, with other indices vanishing, and the combinatorial factors are $3,6,6,3,3$, respectively.) (ii) restricts the sum to terms with $n$ factors, and (i) produces the correct weight $l$. These restrictions are solved by

$$
\left(i^{\prime}\right): i_{1}=l-\sum_{j=2}^{l} j i_{j} \quad \text { and } \quad\left(i i^{\prime}\right): i_{0}=n-i_{1}-\sum_{j=2}^{l} i_{j}=n-l+\sum_{j=2}^{l}(j-1) i_{j} .
$$

From $i_{1} \geq 0$, i.e., $l-\sum_{j=2}^{l} j i_{j} \geq 0$, one infers $i_{2} \leq\left\lfloor\frac{l}{2}\right\rfloor$; thus, $i_{2} \in\left[0,\left\lfloor\frac{l}{2}\right\rfloor\right]$. For given $i_{2}$ in this range, $i_{3} \leq\left\lfloor\frac{l-2 i_{2}}{2}\right\rfloor$, etc. In general,

$$
0 \leq i_{k} \leq\left\lfloor\left(l-\sum_{j=2}^{k-1} j i_{j}\right) k\right\rfloor \text { for } k=2,3, \ldots, l
$$

with the sum replaced by zero for $k=2$. This accounts for the upper boundaries $\left\lfloor b_{k}\right\rfloor$ in (30). Now, because $i_{0} \geq 0$, (ii') implies a lower bound for $i_{l}$, the index of the last sum, viz

$$
i_{l} \geq\left\lceil\left(l-n-\sum_{j=2}^{l-1}(j-1) i_{j}\right) /(l-1)\right\rceil
$$

with the ceiling function $\Gamma \cdot 7$. In any case $i_{l} \geq 0$; therefore, the lower boundary for the $i_{l}$-sum is $a_{l}$ as given in (31). All restrictions have then been solved and the lower boundaries of the other sums are given by $a_{k}=0$ for $k=2, \ldots, l-1$. As to the combinatorial factor, it now depends only on $n, l,\left\{i_{j}\right\}_{2}^{l}$ and is written as $\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle$. It counts the number of possibilities for the occurrence of the considered term of the sum which is given by

$$
\binom{n}{i_{0}}\binom{n-i_{0}}{i_{1}} \cdots\binom{n-\sum_{j=0}^{l-1} i_{j}}{i_{l}}=n!/\left(\prod_{j=0}^{l} i_{j}!\right)\left(n-\sum_{j=0}^{l} i_{j}\right)!.
$$

Inserting $i_{0}$ and $i_{1}$ from (ii') and ( $i^{\prime}$ ), respectively, remembering (ii), produces $\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle$ as given in (32). Finally, $\sum\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle C_{0}^{i_{0}} C_{1}^{i_{1}} \cdots C_{l}^{i_{l}}$ is transformed into ( $l-1$ ) nested sums with boundaries $a_{k}$ and $\left\lfloor b_{k}\right\rfloor$ after replacement of $i_{1}$ and $i_{0}$. This completes the proof of (29) for the nontrivial $l \geq 2$ cases.

Corollary 1-Catalan convolutions: For Catalan's sequence $\left\{C_{n}\right\}_{0}^{\infty}$, the $n^{\text {th }}$ convolution sequence for $n \in \mathbf{N}$ is given by $C_{0}(n)=1, C_{1}(n)=n$ and, for $l=2,3, \ldots$, by

$$
\begin{equation*}
C_{l}(n)=\left(\prod_{k=2}^{l} \sum_{i_{k}=a_{k}}^{\left\llcorner b_{k}\right\rfloor}\right)\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle \prod_{j=2}^{l}\left(\frac{C_{j}^{i_{j}}}{i_{j}!}\right), \tag{35}
\end{equation*}
$$

with (30), (31), and (32).
Proof: This is Lemma 1 with $C_{0}=1=C_{1}$.
Example 1: $C_{4}(3)=3 C_{4}+5 C_{3}+3 C_{2}^{2}+3 C_{2}=90$.
Corollary 2: With the Catalan generating function $c(x)$ and the definition, one has, for $n \in \mathbf{N}$, $c^{-n}(x)=: \sum_{l=0}^{\infty} C_{l}(-n) x^{l}$, for $l=2,3, \ldots$,

$$
\begin{equation*}
C_{l}(-n)=(-1)^{l}\left(\prod_{k=2}^{l} \sum_{i_{k}=a_{k}}^{\left\lfloor b_{k}\right\rfloor} \frac{(-1)^{(k-1) i_{k}}}{i_{k}!}\right)\left\langle n, l,\left\{i_{j}\right\}_{2}^{l}\right\rangle \prod_{j=2}^{l-1} C_{j}^{i_{j+1}} \tag{36}
\end{equation*}
$$

with (30), (31), (32), and Catalan numbers $C_{k}$. In addition, $C_{0}(-n)=1, C_{1}(-n)=-n$.
Proof: Lemma 1 is used for powers of $c(x)$ replaced by those of $c^{-1}(x)=1-x c(x)$, with the Catalan generating function $c(x)$. Hence, $c^{-1}(x)=\sum_{k=0}^{\infty} C_{k}(-1) x^{k}$ with

$$
C_{k}(-1)=\left\{\begin{array}{ll}
1 & \text { for } k=0, \\
-C_{k-1} & \text { for } k=1,2, \ldots
\end{array} \quad \text { Then, in Lemma } 1, C_{k} \text { is replaced by } C_{k}(-1)\right.
$$

Example 2: $C_{4}(-3)=-3 C_{3}+6 C_{2}-3+3=-3$.
Convolutions of Catalan's sequence have been encountered in various contexts, for example, in the enumeration of nonintersecting path pairs on a square lattice (see [12], [18], [5]), and in the problem of inverting triangular matrices with Pascal triangle entries (see [6] and earlier works cited there; they also appear in [15], p. 148).
Note 4: Shapiro's Catalan triangle has entries

$$
B_{n, k}=\frac{k}{n}\binom{2 n}{n-k} \text { for } n \geq k \geq 1, \text { and } B_{n, k}=\left[x^{n}\right]\left(x^{k} \hat{c}^{k}(x)\right)
$$

with $\left[x^{n}\right] f(x)$ denoting the coefficient of $x^{n}$ in the expansion of $f(x)$ around $x=0$. In this case, $\hat{c}(x)=(c(x)-1) / x=c^{2}(x)$. (See [12], Propositions (2.1) and (3.3), with $i_{j} \in \mathbf{N}$, not $\mathbf{N}_{0}$.) In [18] this triangle of numbers from [12] reappears as $b(n, k)$, and it is shown there that $B_{n, k} \equiv b(n, k)=$ $\left[x^{n}\right]\left(x c^{2}(x)\right)^{k}$, in accordance with the identity $\hat{c}(x)=c^{2}(x)$. Therefore, only even powers of $c(x)$ appear in Shapiro's Catalan triangle. In [5], $C_{l}(n)$ appears as special case ${ }_{2} d_{2-n, l+1}$. In [6], all powers of $c(x)$ show up as convolutions for the special case of the $S_{1}$ sequence there. The entries of the $S_{1}$-array ([6], p. 397) are $\left[x^{n}\right] c^{k+1}(x)$ for $n, k \in \mathbf{N}_{0}$.

The anonymous referee of this paper noticed that the inverse of the lower triangular matrix $S_{n, k}=\left[x^{k}\right] S_{n}(x)$, for $n, k \in \mathbf{N}_{0}$, with Chebyshev's $S_{n}(x)=U_{n}(x / 2)$ polynomials is the lower triangular convolution matrix obtained from its first $(k=0)$ column sequence generated by $c\left(x^{2}\right)$ (Catalan numbers alternating with zeros). This follows from the fact that the $\mathbf{S}$-matrix is also a lower triangular convolution matrix with generating function $1 /\left(1+x^{2}\right)$ of its first column. See [13] for such type of matrices $\mathbf{M}$ and the relation between the generating functions of the first columns of $\mathbf{M}$ and $\mathbf{M}^{-1}$. The head of this Catalan triangle can be viewed under number A053121 in the on-line database [14]. See also [6] for inverses of Pascal-type arrays.

## Lemma 2-Explicit form of Catalan convolutions [12], [18], [6], [4], [11], and [5]:

For $n \in \mathbb{R}, l \in \mathbf{N}_{0}$ :

$$
\begin{equation*}
C_{l}(n)=\frac{n}{l}\binom{2 l+n-1}{l-1}=\frac{n}{n+2 l}\binom{n+2 l}{l}=\frac{n}{l+n}\binom{2 l+n-1}{l} \tag{37}
\end{equation*}
$$

Proof: Three equivalent expressions have been given for convenience. See [4], page 201, equation (5.60), with $\mathscr{B}_{2}(z)=c(z), t \rightarrow 2, k \rightarrow l, r \rightarrow n$. The proof of (5.60) appears as (7.69) on page 349 of [4], with $m=2, n=l \in \mathbb{R}$.

The same formula occurs as Exercise 213 in Vol. 1 of [11] for $\beta=2$ as a special instance of Exercises 211 and 212. Put $\alpha=n$ and $n=l$ in the solution of Exercise 213 on page 301.

In order to prove this lemma from [12] or [18], one can use

$$
C_{l}(n)=\sum_{j=0}^{\min (l, n)}\binom{n}{j} \hat{C}_{l}(j)
$$

obtained from $c(x)=: 1+\hat{c}(x)$ with

$$
\hat{c}^{n}(x)=: \sum_{k=n}^{\infty} \hat{C}_{k}(n) x^{k-n}
$$

The result in [12] and [18] is, with this notation,

$$
\hat{C}_{l}(j)=B_{l, j}=b(l, j)=\frac{j}{l}\binom{2 l}{l-j}
$$

Inserting this in the given sum, making use of the identity $j\binom{n}{j}=n\binom{n-1}{j-1}$ and the Vandermonde convolution identity, leads to Lemma 2 at least for positive integer $n$, but one can continue this formula to real (or complex) $n$.

In [6], one finds this result as equation (3.1), page 402, for $i=1: s_{1}(l, n)=C_{l}(n)$.
In [5], ${ }_{2} d_{2-n, l+1}=C_{l}(n)$, with the result given in Theorem 2.3, equation (2.6), page 71 .
Note 5: As a side remark we mention that, from (37), $E_{l}(x):=l!C_{l}(x)$ (with real $n=x$ ) is a polynomial of degree $l$, viz $\prod_{j=0}^{l-1}(x+l+1+j)$. These polynomials, which are not the subject of this work, are known (see [8] and references given there) as exponential convolution polynomials satisfying $E_{l}(x+y)=\sum_{k=0}^{l}\binom{l}{k} E_{k}(x) E_{l-k}(y)$.

We now compute the coefficients $C_{l}(n)=\left[x^{l}\right] c^{n}(x)$ (see Note 4 for this notation) from our formula given in Proposition 1. This can be done for $n \in \mathbb{Z}$.

First, consider $n \in \mathbf{N}_{0}$. For $n=0$ and $n=1$, there is nothing new due to the inputs $S_{-2}=-1$, $S_{-1}=0$, and $S_{0}=1 . C_{l}(n)=0$ for negative integer $l$. Therefore, terms proportional to $1 / x^{l}$ with
$l \in \mathbf{N}$ have to cancel in (23), or in (1). For $n=2,3, \ldots$, terms of the type $1 / x^{n-j}$ occur for $j \in$ $\left\{1,2, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$. The coefficient of $1 / x^{n-j}$ in $p_{n-1}(x)$ is $(-1)^{j}\binom{n-1-j}{j-1}$ (see Note 1 for the explicit form of $\left.p_{n-1}\right)$. For the $1 / x^{n-j}$ coefficient in $q_{n-1}(x) c(x)$, one finds the convolution

$$
\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-(j-l)}{j-l-1} C_{l} .
$$

Compensation of both coefficients leads to identity ( $P 1$ ) given in (4) after $(j-1)$ has been traded for $p$. Thus, after a shift $n \rightarrow n+2$,

Proposition 2-Identity (P1): For $n \in \mathbf{N}_{0}$ and $p=0,1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor$, identity ( $P 1$ ), given in equation (4), holds.

Example 3: $n=2(k-1), p=k-1$, and $n=2 k-1, p=k-1$ for $k \in \mathbf{N}$;

$$
\sum_{l=0}^{k-1}(-1)^{l}\binom{k+l}{2 l+1} C_{l}=1, \quad \sum_{l=0}^{k-1}(-1)^{l}\binom{k+l+1}{2(l+1)} C_{l}=k ;
$$

e.g., $k=3: 3 C_{0}-4 C_{1}+1 C_{2}=1,6 C_{0}-5 C_{1}+1 C_{2}=3$.

For $n=2,3, \ldots$, terms in (1), or in (23), proportional to $x^{k}$ with $k \in \mathbf{N}_{0}$ arise only from $q_{n-1}(x) c(x)$, and they are given by the convolution (cf. Note 1),

$$
\sum_{l=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{l}\binom{n-1-l}{l} C_{k+n-1-l} .
$$

For $n=1$, this is $C_{k}$. The left-hand side of $(1)$ contributes $C_{k}(n)$, and $C_{k}(1)=C_{k}$. Therefore,
Proposition 3-Identity (P3): For $n \in \mathbf{N}, k \in \mathbf{N}_{0}$, identity (P2), given in equation (5) with (3) holds.

Example 4: $k=0,(n-1) \rightarrow n$ :

$$
\sum_{l=1}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^{l+1}\binom{n-l}{l} C_{n-l}=C_{n}-1
$$

e.g., $n=3: 2 C_{2}=C_{3}-1, n=4: 3 C_{3}-1 C_{2}=C_{4}-1$.

Now consider negative powers in (1), i.e., $c^{-n}(x), n \in \mathbf{N}$. No negative powers of $x$ appear (cf. Note 1 for the explicit form of $p_{-(n+1)}(x)$ and $\left.q_{-(n+1)}(x)\right)$. The coefficient of $x^{k}, k \in \mathbf{N}_{0}$, of the right-hand side of (1) is

$$
(-1)^{k}\binom{n-k}{k}-\sum_{l=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^{l}\binom{n-1-l}{l} C_{k-1-l}
$$

where the first term, arising from $p_{-(n+1)}(x)$, contributes only for $k \in\left\{0,1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$. In the summation, one also needs $l \leq k-1$ because no Catalan numbers with negative index occur in (1). The left-hand side of (1) has $\left[x^{k}\right] c^{-n}(x)=C_{k}(-n)$. From the last equation in (37), one finds

$$
C_{k}(-n)=\frac{n}{n-k}\binom{2 k-n-1}{k}=(-1)^{k} \frac{n}{n-k}\binom{n-k}{k} .
$$

In the last equation, the upper index in the binomial has been negated (cf. [4], (5.14)). Two sets of identities follow, depending on the range of $k$.

Proposition 4-Identity (P3): For $n \in \mathbf{N}, k \in\left\{0,1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$, identity ( $P 3$ ), given in equation (6), holds.
Example 5: $k=3, n \geq 6: C_{2}-(n-2) C_{1}+\binom{n-3}{2} C_{0}=\binom{n-4}{2}$.
Proposition 5-Identity (P4): For $n \in \mathbf{N}, k \in \mathbf{N}$, with $k \geq\left\lfloor\frac{n}{2}\right\rfloor+1$, identity (P4), given in equation (7), holds.

In (P4), only the $q_{-(n+1)}(x) c(x)$ part of (1) contributed, and we used the first expression for $C_{k}(-n)$ in (37). In (P3), where $p_{-(n+1)}(x)$ also contributed, we used the negated binomial coefficient for $C_{l}(-n)$ and absorption in the resulting one.

Note that (37) implies $C_{k}(-n)=-C_{k-n}(n)$ for $k, n \in \mathbf{N}, k \geq n$, and $C_{k}(0)=\delta_{k, 0}$.
Example 6: $n=5, k \geq 3$ :

$$
C_{k-1}-3 C_{k-2}+C_{k-3}=\frac{5}{k}\binom{2 k-6}{k-1} ;
$$

e.g., $k=7: C_{6}-3 C_{5}+C_{4}=20$.

If one uses the binomial formula for $c^{-n}(x)=(1-x c(x))^{n}$ and $c^{n}(x)=\sum_{k=0}^{\infty} C_{k}(n) x^{k}$, one arrives at equation (8).

## 3. SOME FAMILIES OF INTEGER SEQUENCES

In this section we present some sequences of positive integers which are defined with the help of the $u_{n}$ polynomials (10).

$$
\begin{equation*}
u_{n}(m):=u_{n}(1 / m)=(\sqrt{m})^{n} S_{n}(\sqrt{m}) \tag{38}
\end{equation*}
$$

The last equation is due to (21). It will be shown that $u_{n}(m)$ is a nonnegative integer for each $m=4,5, \ldots$ and $n=-1,0, \ldots$. Also negative integers $-m, m \in \mathbf{N}$ are of interest. In this case, we add a sign factor:

$$
\begin{equation*}
v_{n}(m):=(-1)^{n} u_{n}(-1 / m)=(-i \sqrt{m})^{n} S_{n}(i \sqrt{m}) \tag{39}
\end{equation*}
$$

From the $S_{n}$ recursion relation (15), one infers those for the $u_{n}(m)$ and $v_{n}(m)$ sequences:

$$
\begin{array}{ll}
u_{n}(m)=m\left(u_{n-1}(m)-u_{n-2}(m)\right), & u_{-1}(m) \equiv 0, \\
v_{0}(m) \equiv 1 ;  \tag{41}\\
v_{n}(m)=m\left(v_{n-1}(m)+v_{n-2}(m)\right), & v_{-1}(m) \equiv 0, \\
v_{0}(m) \equiv 1 .
\end{array}
$$

This shows that $v_{n}(m)$ constitutes a nonnegative integer sequence for positive integer $m$. It describes certain generalized Fibonacci sequences (see, e.g., [7] with $v_{n}(m)=W_{n+1}(0,1 ; m, m)$ ). For example, $v_{n}(m)$ counts the length of the binary word $W(m ; n)$ obtained at step $n$ from the substitution rule $1 \rightarrow 1^{m} 0,0 \rightarrow 1^{m}$, starting at step $n=0$ with 0 . The number of 1 's, resp. 0 's, in $W(m ; n)$ is $m v_{n-1}(m)$, resp. $m v_{n-2}(m)$. E.g., $W(2 ; 3)=(110)^{2} 1^{2}(110)^{2} 1^{2}$ and $v_{3}(2)=16,2 v_{2}(2)=12$, and $2 v_{1}(2)=4$. For $m=1$, this substitution rule produces the well-known Fibonacci-tree. Of course, one can define in a similar manner generalized Lucas sequences using the polynomials $\left\{v_{n}\right\}$ given in (11). Each $u_{n}(m)$ sequence (which is identified with $W_{n+1}(0,1 ; m,-m)$ of [7]) turns
out to be composed of two simpler sequences, viz $u_{2 k}(m)=: m^{k} \alpha_{k}(m)$ and $u_{2 k-1}(m)=: m^{k} \beta_{k}(m)$, $k \in \mathbf{N}_{0}$. These new sequences, which are due to (38), given by $\alpha_{k}=S_{2 k}(\sqrt{m})$ and $\beta_{k}(m)=$ $S_{2 k-1}(\sqrt{m}) / \sqrt{m}$, satisfy therefore the following relations:

$$
\begin{equation*}
\beta_{k+1}(m)=(m-2) \beta_{k}(m)-\beta_{k-1}(m), \quad \beta_{0}(m) \equiv 0, \beta_{1}(m) \equiv 1, \tag{42}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha_{k-1}(m)=\beta_{k}(m)+\beta_{k-1}(m) . \tag{43}
\end{equation*}
$$

From (42) it is now clear that $\beta_{n}(m)$ is a nonnegative integer sequence for $m=4,5, \ldots$. (In [7], $\beta_{n}(m)=W_{n}(0,1 ; m-2,-1)$.) This property is then inherited by the $\alpha_{n}(m)$ sequences due to (43), and then by the composed sequence $u_{n}(m)$.

The ordinary generating functions are:

$$
\begin{align*}
& g_{\beta}(m ; x):=\sum_{n=0}^{\infty} \beta_{n}(m) x^{n}=\frac{x}{x^{2}-(m-2) x+1}, \quad g_{\alpha}(m ; x):=\sum_{n=0}^{\infty} \alpha_{n}(m) x^{n}=\frac{1+x}{x^{2}-(m-2) x+1}  \tag{44}\\
& g_{u}(m ; x):=\sum_{n=0}^{\infty} u_{n}(m) x^{n}=\frac{1}{1-m x+m x^{2}}, \quad g_{v}(m ; x):=\sum_{n=0}^{\infty} v_{n}(m) x^{n}=\frac{1}{1-m x-m x^{2}} \tag{45}
\end{align*}
$$

Note 6: The $\left\{\beta_{n}(m)\right\}$ sequences for $m=4,5,6,7,8,10$ appear in the book [14]. The case $m=4$ produces the sequence of nonnegative integers; $m=5$ are the even-indexed Fibonacci numbers. The $m=9$ sequence appears in Sloane's "On-Line-Encyclopedia" [14] as A004187. The $\left\{\alpha_{n}(m)\right\}$ sequences for $m=4,5,6,8$ also appear in the book [14]. The case $m=4$ yields the positive odd integer sequence; $m=5$ is the odd-indexed Lucas number sequence. The $m=7$ sequence appears in the database [14] as A030221. The composed sequences $\left\{u_{n}(m)\right\}$ do not appear in the book [14], but some of them are found in the database [14]. $m=4$ is the sequence $(n+1) 2^{n}, \mathrm{~A} 001787$, and $m=5,6,7$ appear as A030191, A030192, and A030140, respectively. As mentioned above, $\left\{v_{n-1}(1)\right\}$ is the Fibonacci sequence. The instances $m=2$ and 3 appear as A002605 and A030195, respectively, in the database [14].

## ACKNOWLEDGMENTS

The author is grateful to Dr. Stephen Bedding for a collaboration on powers of matrices. In Section 2 a result for $2 \times 2$ matrices (here T) was recovered. The anonymous referee of this paper asked for a combinatorial interpretation of the $v_{n}(m)$ numbers, pointed out references [3], [13], [15], [17], and noticed that the inverse of the coefficient matrix for Chebyshev's $S$ polynomials furnishes a Catalan triangle (see Note 4).

## REFERENCES

1. M. Abramowitz \& I. A. Stegun. Handbook of Mathematical Functions. New York: Dover, 1968.
2. M. Gardner. Time Travel and Other Mathematical Bewilderments. Chapter 20. New York: W. H. Freeman, 1988.
3. C. D. Godsil. Algebraic Combinatorics. New York and London: Chapman \& Hall, 1993.
4. R. L. Graham, D. E. Knuth, \& O. Patashnik. Concrete Mathematics. Reading, MA: AddisonWesley, 1989.
5. P. Hilton \& J. Pedersen. "Catalan Numbers, Their Generalizations, and Their Uses." The Mathematical Intelligencer 13 (1991):64-75.
6. V. E. Hoggatt, Jr., \& M. Bicknell. "Catalan and Related Sequences Arising from Inverses of Pascal's Triangle Matrices." The Fibonacci Quarterly 14.5 (1976):395-405.
7. A. F. Horadam. "Special Properties of the Sequence $W_{n}(a, b ; p, q)$." The Fibonacci Quarterly 5.5 (1967):424-434.
8. D. E. Knuth. "Convolution Polynomials." The Mathematica J. 2.1 (1992):67-78.
9. W. Lang. "On Sums of Powers of Zeros of Polynomials." J. Comp. and Appl. Math. 89 (1998):237-56.
10. M. Petkovšek, H. S. Wilf, \& D. Zeilberger. $A=B$. Wellesley, MA: A. K. Peters, 1996.
11. G. Pólya \& G. Szegö. Aufgaben und Lehrsätze aus der Analysis I. 4th ed. Berlin: Springer, 1970.
12. L. W. Shapiro. "A Catalan Triangle." Discrete Math. 14 (1976):83-90.
13. L. W. Shapiro, S. Getu, W.-J. Woan, \& L. C. Woodson. "The Riordan Group." Discrete Appl. Math. 34 (1991):229-39.
14، N. J. A. Sloane \& S. Plouffe. The Encyclopedia of Integer Sequences. San Diego, CA: Academic Press, 1995; see also N. J. A. Sloane's "On-Line Encyclopedia of Integer Sequences," http//:www.research.att.com/njas/sequences/index.html.
14. D. R. Snow. "Spreadsheets, Power Series, Generating Functions, and Integers." The College Math. J. 20 (1989):143-52.
15. R. P. Stanley. Enumerative Combinatorics. Vol. 2. Cambridge, MA: Cambridge University Press, 1999; excerpt "Problems on Catalan and Related Numbers," available from http/www. math.mit-edu/ $/$ rstan/ec/ec.html.
16. G. Viennot. "Une théorie combinatoire des polynômes orthogonaux generaux." Notes de conférences donnée au Département de mathématique et d'informatique, Université du Québec à Montreal, Septembre-Octobre, 1983.
17. W.-J. Woan, L. Shapiro, \& D. G. Rogers. "The Catalan Numbers, the Lebesgue Integral, and $4^{n-2}$." Amer. Math. Monthly 101 (1997):926-31.
AMS Classification Numbers: 11B83, 11B37, 33C45
