ON AN OBSERVATION OF D'OCAGNE CONCERNING THE FUNDAMENTAL SEQUENCE

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(*Submitted December 1998-Final Revision April 1998*)

1. INTRODUCTION

Following the notation in [3], we consider the sequence $\{W_n\} = \{W_n(a, b; p, q)\}$ defined, for all integers n, by

$$W_n = pW_{n-1} - qW_{n-2},\ W_0 = a,\ W_1 = b. \tag{1.1}$$

Throughout this paper we take a, b, p, and q to be arbitrary integers with $q \neq 0$.

Distinguished among all the sequences generated by the recurrence in (1.1) is the pair $U_n = W_n(0, 1; p, q)$ and $V_n = W_n(2, p; p, q)$, whose importance was first recognized by Lucas [4]. The sequences $\{U_n\}$ and $\{V_n\}$ are often referred to as the *fundamental* and *primordial* sequences, respectively [13]. Because of their special properties, $\{U_n\}$ and $\{V_n\}$ continue to be the focus of much attention [2], [5], [9], [12]. Our interest in this paper is in a property of $\{U_n\}$ which, according to Dickson ([1], p. 409), was first observed by D'Ocagne. D'Ocagne observed that there exist **integers** c_0 and c_1, independent of n, such that

$$W_n = c_0U_n + c_1U_{n+1}, \quad n \in \mathbf{Z}. \tag{1.2}$$

Indeed, it can be proved by induction that

$$W_n = (W_1 - pW_0)U_n + W_0U_{n+1}, \quad n \in \mathbf{Z}. \tag{1.3}$$

In this sense $\{U_n\}$ can be regarded as a "basis" for the sequences generated by the recurrence in (1.1). In fact, as stated in the reference of Dickson mentioned above, D'Ocagne observed this property for the higher-order analogs of $\{U_n\}$.

It is natural to ask if there are other sequences generated by the recurrence in (1.1) which also possess this property of $\{U_n\}$. To be more precise, we make the following definition.

Property of D'Ocagne: An integer sequence $\{S_n\} = \{W_n(S_0, S_1; p, q)\}$ is said to have the property of D'Ocagne if there exist integers c_0 and c_1, independent of n, such that $W_n = c_0S_n + c_1S_{n+1}$, $n \in \mathbf{Z}$.

For $q = \pm 1$ we have characterized all sequences which have the property of D'Ocagne. The object of this paper is to present our results.

2. PRELIMINARY RESULTS

For the remainder of the paper we take $\{S_n\} = \{W_n(S_0, S_1; p, q)\}$ to be an integer sequence. In order to make the paper self-contained, we now list several known results which will be required in the sequel.

Lemma 1:

$$D_n = \begin{vmatrix} W_n & S_n & S_{n+1} \\ W_1 & S_1 & S_2 \\ W_0 & S_0 & S_1 \end{vmatrix} = 0, \quad n \in \mathbf{Z}.$$

Lemma 2: The points with integer coordinates on the conics $y^2 - 3xy + x^2 = \pm 1$ are precisely the pairs $(x, y) = \pm(F_n, F_{n+2})$.

Lemma 3: In (1.1) suppose $p \neq 0$ and $q = -1$. Then the points with integer coordinates on the conics $y^2 - pxy - x^2 = \pm 1$ are precisely the pairs $(x, y) = \pm(U_n, U_{n+1})$.

Lemma 4: In (1.1) suppose $|p| > 2$ and $q = 1$. Then the points with integer coordinates on the conic $y^2 - pxy + x^2 = 1$ are precisely the pairs $(x, y) = \pm(U_n, U_{n+1})$.

Lemma 1 is a special case of Theorem 1 in [7]. Lemmas 2, 3, and 4 are special cases of Theorems 1, 2, and 5, respectively, in [6].

We also require several well-known theorems concerning the integer solutions of the Pell equation

$$x^2 - dy^2 = 1, \tag{2.1}$$

and its generalization

$$x^2 - dy^2 = N. \tag{2.2}$$

Here we assume that d is a positive integer that is not a perfect square and N is an integer.

Theorem 1 (see Theorem 11.5 in [11]): Let h_m / k_m denote the m^{th} convergent of the simple continued fraction of $\sqrt{d}$, $m = 0, 1, 2, \ldots$, and let l be the period length of this continued fraction. If l is even, then $(x, y) = (h_{l-1}, k_{l-1})$ is a solution of (2.1).

Theorem 2 (see Theorem 11.3 in [11]): Suppose $|N| < \sqrt{d}$. If (x, y), with x and y positive, is a solution of (2.2), then x / y is a convergent of the simple continued fraction of $\sqrt{d}$.

Theorem 3 (see Theorem 3.3, p. 128, in [10]): If (2.2) has a solution, then it has infinitely many solutions. At least one of these solutions satisfies

$$0 < x < \sqrt{((x_0 + 1)|N|/2)},$$

where (x_0, y_0) is the fundamental solution of (2.1).

Finally, we require the following lemma. For part (a), see page 389 of [11]. Indeed, both parts can be established with the use of the standard method for developing a surd as a continued fraction. See, for example, page 176 of [8].

Lemma 5: Let d be a positive integer.

(a) If $d > 3$ is odd, the simple continued fraction of $\sqrt{d^2 - 4}$ is

$$\left[d - 1; \overline{1, (d-3)/2, 2, (d-3)/2, 1, 2d - 2}\right].$$

(b) If $d > 4$ is even, the simple continued fraction of $\sqrt{d^2 - 4}$ is

$$\left[d - 1; \overline{1, (d-4)/2, 1, 2d - 2}\right].$$

3. THE MAIN RESULTS

Our first theorem gives necessary and sufficient conditions for the sequence $\{S_n\}$ to have the property of D'Ocagne.

Theorem 4: Suppose $S_1^2 - S_0S_2 \neq 0$. Then $\{S_n\}$ has the property of D'Ocagne if and only if $S_1^2 - S_0S_2 = \pm 1$.

Proof: From Lemma 1 we have

$$(S_1^2 - S_0S_2)W_n = (S_1W_1 - S_2W_0)S_n + (S_1W_0 - S_0W_1)S_{n+1}, \quad n \in \mathbf{Z}. \tag{3.1}$$

Hence, if $S_1^2 - S_0S_2 = \pm 1$, then $\{S_n\}$ has the property of D'Ocagne.

Conversely, suppose $\{S_n\}$ has the property of D'Ocagne. Then there exist integers c_0 and c_1 such that

$$U_n = c_0S_n + c_1S_{n+1}, \quad n \in \mathbf{Z}. \tag{3.2}$$

Putting $n = 0$ and $n = 1$, we see from Cramer's rule that c_0 and c_1 are unique. Now, by (3.1), we have

$$U_n = \frac{S_1}{S_1^2 - S_0S_2}S_n - \frac{S_0}{S_1^2 - S_0S_2}S_{n+1}, \quad n \in \mathbf{Z}. \tag{3.3}$$

But, by the uniqueness of c_0 and c_1 we have

$$c_0 = \frac{S_1}{S_1^2 - S_0S_2} \quad \text{and} \quad c_1 = -\frac{S_0}{S_1^2 - S_0S_2},$$

which means that $S_1^2 - S_0S_2$ divides S_n, $n \geq 0$. Consequently, putting $n = 1$ in (3.2), we see that $S_1^2 - S_0S_2$ divides 1, and this completes the proof. □

Our next theorem characterizes those sequences $\{S_n\} = \{W_n(S_0, S_1; p, -1)\}$ that have the property of D'Ocagne.

Theorem 5: If $p \neq 0$, then $\{S_n\} = \{W_n(S_0, S_1; p, -1)\}$ has the property of D'Ocagne if and only if $(S_0, S_1) = \pm(U_m, U_{m+1})$ for some integer m.

Proof: We first prove that $S_1^2 - S_0S_2 \neq 0$. On the contrary, suppose $S_1^2 - S_0S_2 = 0$. If one of S_0, S_1, or S_2 is zero, one of the others must be zero, which means that $\{S_n\}$ is the zero sequence. So we can assume that $S_0S_1S_2 \neq 0$. Now

$$\frac{S_1}{S_0} = \frac{S_2}{S_1} = \frac{pS_1 + S_0}{S_1} = p + \frac{S_0}{S_1},$$

and this implies that

$$\frac{S_1}{S_0} = \frac{p \pm \sqrt{p^2 + 4}}{2}.$$

But since $p^2 + 4$ is not a perfect square, S_1 / S_0 is irrational, which is a contradiction. Hence, $S_1^2 - S_0S_2 \neq 0$. Then, by Theorem 4, $\{S_n\}$ has the property of D'Ocagne if and only if $S_1^2 - S_0S_2 = S_1^2 - pS_0S_1 - S_0^2 = \pm 1$. Theorem 5 now follows from Lemma 3. □

Our final theorem characterizes those sequences $\{S_n\} = \{W_n(S_0, S_1; p, 1)\}$ that have the property of D'Ocagne.

Theorem 6: Let $|p| > 2$ and let $\{S_n\} = \{W_n(S_0, S_1; p, 1)\}$.

(a) If $p = 3$, then $\{S_n\}$ has the property of D'Ocagne if and only if $(S_0, S_1) = \pm(F_m, F_{m+2})$ for some integer m.

(b) If $p=-3$, then $\{S_n\}$ has the property of D'Ocagne if and only if $(S_0, S_1)=\pm(F_m, -F_{m+2})$ for some integer m.

(c) If $|p|>3$, then $\{S_n\}$ has the property of D'Ocagne if and only if $(S_0, S_1)=\pm(U_m, U_{m+1})$ for some integer m.

Proof: As in the proof of Theorem 5, it is straightforward to prove that $S_1^2-S_0S_2\neq 0$. Since $S_1^2-S_0S_2-S_1^2-pS_0S_1+S_0^2$, we see from Theorem 4 that $\{S_n\}$ has the property of D'Ocagne if and only if

$$S_1^2-pS_0S_1+S_0^2=\pm 1. \tag{3.4}$$

Now part (a) follows immediately from Lemma 2. Writing $S_1^2+3S_0S_1+S_0^2$ as $(-S_1)^2-3S_0(-S_1)+S_0^2$, we see that part (b) also follows from Lemma 2.

To prove part (c), we consider first the equation

$$S_1^2-pS_0S_1+S_0^2=1,\quad |p|>3. \tag{3.5}$$

By Lemma 4, the solutions of (3.5) are precisely the pairs $(S_0, S_1)=\pm(U_m, U_{m+1})$. Next we consider the equation

$$S_1^2-pS_0S_1+S_0^2=-1,\quad |p|>3, \tag{3.6}$$

and solve for S_1 to obtain

$$S_1=\frac{pS_0\pm\sqrt{(p^2-4)S_0^2-4}}{2},\quad |p|>3. \tag{3.7}$$

To complete the proof of (c), it is enough to prove that (3.7) yields no integer pairs (S_0, S_1). We accomplish this by proving that the generalized Pell equation

$$x^2-(p^2-4)y^2=-4,\quad |p|>3, \tag{3.8}$$

has no solutions. It suffices to consider only $p>3$.

To begin we assume that p is odd. Using Lemma 5, part (a), we find the convergents h_m/k_m, $0\leq m\leq 5$, of the continued fraction expansion of $\sqrt{p^2-4}$ from the following table. In the table, the a_m are the partial quotients.

TABLE 1

m	a_m	h_m	k_m
0	$p-1$	$p-1$	1
1	1	p	1
2	$(p-3)/2$	$(p^2-p-2)/2$	$(p-1)/2$
3	2	p^2-2	p
4	$(p-3)/2$	$(p^3-2p^2-3p+4)/2$	$(p^2-2p-1)/2$
5	1	$(p^3-3p)/2$	$(p^2-1)/2$

Now by Theorem 1 and Lemma 5, part (a), and as is easily verified by substitution, (h_5, k_5) is a solution of $x^2-(p^2-4)y^2=1$. For integers $x_0\geq 3$, $(x_0-1)^2>3$. This implies $x_0^2>2(x_0+1)$ which in turn implies $x_0>\sqrt{2(x_0+1)}$. Consequently, taking $x_0=(p^3-3p)/2>3$, we can replace the inequality in Theorem 3 by the more generous inequality $0<x<x_0$. But by trial we find that none of the pairs (h_m, k_m), $0\leq m\leq 4$, is a solution of (3.8). Hence, by Theorems 2 and 3, (3.8) has no solutions.

To complete the proof, we consider (3.8) for $p \geq 4$, p even. For $p = 4$, equation (3.8) has no solutions since it has no solutions modulo 3. For $p > 4, p$ even, we use Lemma 5, part (b), to construct the following table for the continued fraction expansion of $\sqrt{p^2-4}$.

TABLE 2

m	a_m	h_m	k_m
0	$p-1$	$p-1$	1
1	1	p	1
2	$(p-4)/2$	$(p^2-2p-2)/2$	$(p-2)/2$
3	1	$(p^2-2)/2$	$p/2$

Now (h_3, k_3) is a solution of $x^2-(p^2-4)y^2 = 1$, but, as is easily verified, none of the pairs (h_m, k_m), $0 \leq m \leq 2$, is a solution of (3.8). Hence, by the same reasoning as before, (3.8) has no solutions for $p > 4, p$ even. This completes the proof of Theorem 6. □

Our attempts to obtain analogs of Theorems 5 and 6 for $q \neq \pm 1$ have, to this point, been unsuccessful. This will continue to be the subject of our endeavors.

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AMS Classification Numbers: 11B39, 11B37