ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-570</u> ***Proposed by H.-J. Seiffert, Berlin, Germany***

Show that, for all positive integers n:

(a)
$$5^{n-1}F_{2n-1} = \sum_{\substack{k=0 \\ 5\nmid 2n-k-1}}^{2n-1} (-1)^k \binom{4n-2}{k};$$

(b)
$$5^{n-1}L_{2n} = \sum_{\substack{k=0 \\ 5\nmid 2n-k}}^{2n} (-1)^{k+1} \binom{4n}{k}.$$

Two closely related identities were given in H-518.

<u>H-571</u> ***Proposed by D. Tsedenbayar, Mongolian Pedagogical University, Warsaw, Poland***

Prove: If $(T_\alpha f)(t) = t^\alpha \int_0^t f(s)\,ds$ with $\alpha \in \mathbf{R}$, then

$$(T_\alpha^n f)(t) = \frac{t^\alpha}{(\alpha+1)^{(n-1)}(n-1)!}\int_0^t (t^{\alpha+1} - s^{\alpha+1})^{n-1} f(s)\,ds, \text{ for } \alpha \neq -1,$$

and

$$(T_\alpha^n f)(t) = \frac{1}{t(n-1)!}\int_0^t \left(\ln\frac{t}{s}\right)^{n-1} f(s)\,ds, \text{ for } \alpha = -1.$$

Remark: If $\alpha = -1$, then T_{-1} is a Cesaro operator; if $\alpha = 0$, then T_0 is a Volterra operator.

A Correction:

<u>H-568</u> ***Proposed by N. Gauthier, Royal Military College of Canada, Kingston, Ontario***

The following was inspired by Paul S. Bruckman's Problem B-871 (Vol. 37, no. 1, February 1999; solved Vol. 38, no. 1, February 2000).

"For integers $n, m \geq 1$, prove or disprove that

$$f_m(n) \equiv \frac{1}{\binom{2n}{n}^2}\sum_{k=0}^{2n}\binom{2n}{k}^2 |n-k|^{2m-1}$$

is the ratio of two polynomials with integer coefficients,

$$f_m(n) = P_m(n) / Q_m(n),$$

where $P_m(n)$ is of degree $\lfloor \frac{3m}{2} \rfloor$ in n and $Q_m(n)$ is of degree $\lfloor \frac{m}{2} \rfloor$; determine $P_m(n)$ and $Q_m(n)$ for $1 \le m \le 5$."

SOLUTIONS

A Piece of Pi

H-558 ***Proposed by Paul S. Bruckman, Berkeley, CA (Vol. 37, no. 4, November 1999)***

Prove the following:

$$\pi = \sum_{n=0}^{\infty} (-1)^n \{6\varepsilon_{10n+1} - 6\varepsilon_{10n+3} - 4\varepsilon_{10n+5} - 6\varepsilon_{10+7} + 6\varepsilon_{10n+9}\}, \text{ where } \varepsilon_m = \alpha^{-m} / m. \qquad (*)$$

Solution by the proposer

Looking at the form of the series expression, it is evidently composed of decisections of the logarithmic series. We begin with the definitions:

$$F_r(z) \equiv \sum_{n=0}^{\infty} z^{10n+r} / (10n+r), \quad r = 1, 2, \ldots, 10, \quad |z| < 1. \qquad (1)$$

Note that $F_r(0) = 0$ and $F_r'(z) = \sum_{n=0}^{\infty} z^{10n+r-1} = z^{r-1} / (1 - z^{10})$. Letting $\theta = \exp(i\pi / 5)$ (a tenth root of unity), we find, using residue theory (or otherwise), that $F_r'(z) = 1/10 \sum_{k=1}^{10} \theta^{-k(r-1)} (1 - x\theta^k)^{-1}$; then, by integration,

$$F_r(z) = -1/10 \sum_{k=1}^{10} \theta^{-kr} \log(1 - x\theta^k). \qquad (2)$$

The following transformation is implemented, valid for all complex $z = re^{i\phi}$:

$$\operatorname{Log} z = \log r + i\phi. \qquad (3)$$

Here, "Log" designates the "principal" logarithm, with $-\pi \le \phi \le \pi$; $r = |z|$, $\phi = \operatorname{Arg} z$. We also note that $2\cos(\pi/5) = \alpha$, $2\cos(2\pi/5) = -\beta = 1/\alpha$, and we let s_j denote $\sin(j\pi/5)$, $j = 1, 2$. We readily find that $2s_1 = (\sqrt{5}/\alpha)^{1/2}$ and $2s_2 = (\alpha\sqrt{5})^{1/2} = 2\alpha s_1$. After a trite but straightforward computation, we obtain the following expressions:

$$F_1(z) = \alpha A(x, \alpha) + \beta A(x, \beta) + B(x) + s_1 C(x) + s_2 D(x), \qquad (4)$$

where

$$A(x, c) = 1/20 \log\{(1 + cx + x^2) / (1 - cx + x^2)\}, \quad B(x) = 1/10 \log\{(1 + x) / (1 - x)\},$$
$$C(x) = 1/5 \tan^{-1}\{2xs_1 / (1 - x^2)\}, \quad D(x) = 1/5 \tan^{-1}\{2xs_2 / (1 - x^2)\};$$

$$F_2(x) = \alpha P(x, \alpha) + \beta P(x, \beta) + Q(x) + s_1 U(x) + s_2 V(x), \qquad (5)$$

where

$$P(x, c) = 1/20 \log(1 + cx^2 + x^4), \quad Q(x) = -1/10 \log(1 - x^2),$$
$$U(x) = 1/5 \tan^{-1}\{2x^2 s_1 / (2 + \alpha x^2)\}, \quad V(x) = 1/5 \tan^{-1}\{2x^2 s_2 / (2 + \beta x^2)\};$$

$$F_3(z) = \beta A(x, \alpha) + \alpha A(x, \beta) + B(x) + s_2 C(x) - s_1 D(x); \qquad (6)$$

$$F_4(x) = \beta P(x, \alpha) + \alpha P(x, \beta) + Q(x) - s_2 U(x) + s_1 V(x); \qquad (7)$$

$$F_5(z) = -2A(x,\alpha) - 2A(x,\beta) + B(x) = 1/10\log\{(1+x^5)/(1-x^5)\}; \quad (8)$$
$$F_6(x) = \beta P(x,\alpha) + \alpha P(x,\beta) + Q(x) + s_2U(x) - s_1V(x); \quad (9)$$
$$F_7(z) = \beta A(x,\alpha) + \alpha A(x,\beta) + B(x) - s_2C(x) + s_1D(x); \quad (10)$$
$$F_8(x) = \alpha P(x,\alpha) + \beta P(x,\beta) + Q(x) - s_1U(x) - s_2V(x); \quad (11)$$
$$F_9(z) = \alpha A(x,\alpha) + \beta A(x,\beta) + B(x) - s_1C(x) - s_2D(x); \quad (12)$$
$$F_{10}(x) = -2P(x,\alpha) - 2P(x,\beta) + Q(x) = -1/10\log(1-x^{10}). \quad (13)$$

Next, we note the following: $F_1(x)+F_3(x)+F_7(x)+F_9(x) = 2A(x,\alpha)+2A(x,\beta)+4B(x)$. Then, using (8):

$$\begin{aligned}G(x) &\equiv 6\{F_1(x)+F_3(x)+F_7(x)+F_9(x)\} - 4F_5(x)\\
&= 12A(x,\alpha)+12A(x,\beta)+24B(x)+8A(x,\alpha)+8A(x,\beta)-4B(x)\\
&= 20\{A(x,\alpha)+A(x,\beta)+B(x)\}\\
&= \log\{\{(1+\alpha x+x^2)(1+\beta x+x^2)(1+x)^2\}/\{(1-\alpha x+x^2)(1-\beta x+x^2)(1-x)^2\}\}\\
&= \log\{\{(1+x+x^2+x^3+x^4)(1+x)^2\}/\{(1-x+x^2+x^3-x^4)/(1-x)^2\}\}\\
&= \log\{(1-x^5)(1+x)^3/[(1-x^5)(1-x)^3]\}.\end{aligned}$$

Thus, $G(ix) = -\log\{(1+ix^5)/(1-ix^5)\} + 3\log\{(1+ix)/(1-ix)\}$, i.e.,

$$6\{F_1(ix)+F_3(ix)+F_7(ix)+F_9(ix)\} - 4F_5(ix) = -2i\tan^{-1}x^5 + 6i\tan^{-1}x. \quad (14)$$

The left side of (14), employing the series definitions, becomes

$$i\sum_{n=0}^{\infty}(-1)^n\{6\varepsilon_{10n+1}(x) - 6\varepsilon_{10n+3}(x) - 4\varepsilon_{10n+5}(x) + 6\varepsilon_{10n+7}(x) - 6\varepsilon_{10n+9}(x)\},$$

where $\varepsilon_m(x) = x^m/m$. We see that, in order to prove the desired identity (*), it suffices to show:

$$-\tan^{-1}(\alpha^{-5}) + 3\tan^{-1}(\alpha^{-1}) = \pi/2. \quad (15)$$

If $\varphi = \tan^{-1}(\alpha^{-1})$, then

$$\tan(3\varphi) = (3\tan\varphi - \tan^3\varphi)/(1-3\tan^2\varphi) = (3\alpha^2-1)/(\alpha^3-3\alpha) = (3\alpha+2)/(1-\alpha) = -\alpha^5.$$

Thus, $3\varphi = \pi - \tan^{-1}(\alpha^5) = \pi - (\pi/2 - \tan^{-1}(\alpha^{-5}))$, which is (15). Q.E.D.

Also solved by R. Martin and H.-J. Seiffert

SUM Formulae

H-559 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 38, no. 1, February 2000)***

Let n and q be nonnegative integers and show that:

a. $$S_n(q) := \sum_{k=1}^{n} \frac{1}{2\cos(2\pi k/n) + (-1)^{q+1}L_{2q}} = \frac{(-1)^{q+1}nL_{qn}}{5F_{2q}F_{qn}};$$

b. $$s_n(q) := \sum_{k=1}^{n} \frac{1}{0.8\sin^2(2\pi k/n) + F_{2q}^2} = \begin{cases} \frac{nL_{2qn}}{F_{2q}L_{2q}F_{qn}L_{qn}}, & n \text{ odd},\\ \frac{nL_{qn}}{F_{2q}L_{2q}F_{qn}}, & n \text{ even}.\end{cases}$$

L_n and F_n are Lucas and Fibonacci numbers.

Solution by H.-J. Seiffert, Berlin, Germany

Let n be a positive integer. Differentiating the known identity [see I.S. Gradshteyn & I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., p. 41, eqn. 1.394 (New York: Acad. Press, 1994)]

$$\prod_{k=1}^{n}(x^2+1-2x\cos(2\pi k/n))=(x^n-1)^2$$

logarithmically gives

$$\sum_{k=1}^{n}\frac{2x-2\cos(2\pi k/n)}{x^2+1-2x\cos(2\pi k/n)}=\frac{2nx^{n-1}}{x^n-1}.$$

Multiplying by x and then subtracting n from both sides of the resulting equation yields

$$\sum_{k=1}^{n}\frac{x^2-1}{x^2+1-2x\cos(2\pi k/n)}=n\frac{x^n+1}{x^n-1}.$$

It now easily follows that

$$\sum_{k=1}^{n}\frac{1}{2\cos(2\pi k/n)-x-1/x}=\frac{nx(x^n+1)}{(1-x^2)(x^n-1)},\tag{1}$$

valid for all real numbers x such that $x\neq 0$ and $x\neq 1$.

Taking $x=(\alpha/\beta)^q$, $q\in Z$, and $q\neq 0$, and using the known Binet forms of the Fibonacci and the Lucas numbers, we easily obtain the desired equation of the first part.

Replacing x by $-x$ in (1) and subtracting the so obtained identity from (1) gives

$$\sum_{k=1}^{n}\frac{2x+2/x}{4\cos^2(2\pi k/n)-(x+1/x)^2}=\frac{nx}{1-x^2}\left(\frac{x^n+1}{x^n-1}+\frac{(-x)^n+1}{(-x)^n-1}\right).$$

Since $\cos^2(2\pi k/n)=1-\sin^2(2\pi k/n)$, after some simple manipulations, we find that

$$T_n(x):=\sum_{k=1}^{n}\frac{1}{0.8\sin^2(2\pi k/n)+0.2(x-1/x)^2}=\frac{nx^2}{0.4(x^4-1)}\left(\frac{x^n+1}{x^n-1}+\frac{(-x)^n+1}{(-x)^n-1}\right),$$

valid for all real numbers x such that $x\neq 0$ and $x\neq 1$. Hence:

$$T_n(x)=\frac{nx^2(x^{2n}+1)}{0.2(x^4-1)(x^{2n}-1)}\ \text{ if } n \text{ is odd;}\quad T_n(x)=\frac{nx^2(x^n+1)}{0.2(x^4-1)(x^n-1)}\ \text{ if } n \text{ is even.}$$

Taking $x=(-\alpha/\beta)^q$, $q\in Z$, and $q\neq 0$, one easily deduces the requested equations of the second part. Q.E.D.

Also solved by P. Bruckman and the proposer.

A Complex Problem

H-560 ***Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 38, no. 1, February 2000)***

Define the sequences of Fibonacci and Lucas polynomials by

$$F_0(x)=0,\ F_1(x)=1,\ F_{n+1}(x)=xF_n(x)+F_{n-1}(x),\ n\in N,$$

and

$$L_0(x)=2,\ L_1(x)=x,\ L_{n+1}(x)=xL_n(x)+L_{n-1}(x),\ n\in N,$$

respectively. Show that, for all complex numbers x and all positive integers n,

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k F_{3k}(x) = F_{2n}(x) + (-x)^n F_n(x)$$

and

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^k L_{3k}(x) = L_{2n}(x) + (-x)^n L_n(x).$$

Solution by Paul S. Bruckman, Berkeley, CA

We begin with the following well-known explicit expressions for $F_n(x)$ and $L_n(x)$, namely,

$$F_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta), \ L_n(x) = \alpha^n + \beta^n, \ n = 0, 1, 2, \ldots, \tag{1}$$

where

$$\alpha = \alpha(x) = (x+\theta)/2, \ \beta = \beta(x) = (x-\theta)/2, \tag{2}$$

$$\theta = \theta(x) = (x^2+4)^{1/2} = \alpha - \beta. \tag{3}$$

Next, we make the following definitions:

$$G_n(y) = \sum_{k=0}^{[n/2]} n/(n-k) \cdot {}_{n-k}C_k \cdot y^k, \tag{4}$$

where ${}_rC_s$ is the combinatorial symbol commonly known as "r choose s," i.e., $\binom{r}{s}$.

Then, if $U_n(x)$ and $V_n(x)$ denote the first and second sum expressions, respectively, given in the statement of the problem, we obtain

$$U_n(x) = \theta^{-1} \sum_{k=0}^{[n/2]} n/(n-k) \cdot {}_{n-k}C_k \cdot x^k(\alpha^{3k} - \beta^{3k}), \text{ or}$$

$$U_n(x) = \theta^{-1}(G_n(\alpha^3 x) - G_n(\beta^3 x)). \tag{5}$$

Similarly,

$$V_n(x) = G_n(\alpha^3 x) + G_n(\beta^3 x), \tag{6}$$

where we also make the following definitions: $U_0(x) = 0$, $V_0(x) = 2$.

Next, we form the following generating functions:

$$R(z, x) = \sum_{n=0}^{\infty} U_n(x) z^n, \quad S(z, x) = \sum_{n=0}^{\infty} V_n(x) z^n, \tag{7}$$

$$T(z, y) = \sum_{n=0}^{\infty} G_n(y) z^n. \tag{8}$$

We see that

$$\begin{aligned} R(z, x) &= \theta^{-1}\{T(z, x\alpha^3) - T(z, x\beta^3)\}, \\ S(z, x) &= T(z, x\alpha^3) + T(z, x\beta^3). \end{aligned} \tag{9}$$

We obtain a closed form expression for $T(z, y)$ as follows:

$$\begin{aligned} T(z, y) &= \sum_{n,k=0}^{\infty} (n+2k)/(n+k)_{n+k}C_k z^{n+2k} y^k \\ &= \sum_{n,k=0}^{\infty} {}_{n+k}C_k z^n (z^2 y)^k + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} {}_{n+k-1}C_{k-1} z^n (z^2 y)^k \end{aligned}$$

$$= (1+z^2y)\sum_{n,\,k=0}^{\infty} {}_{n+k}C_k z^n (z^2y)^k = (1+z^2y)\sum_{n,\,k=0}^{\infty} {}_{-n-1}C_k z^n (-z^2y)^k$$

$$= (1+z^2y)\sum_{n=0}^{\infty} z^n (1-z^2y)^{-n-1} = (1+z^2y)(1-z^2y)^{-1}\{1-z/(1-z^2y)\}^{-1},$$

or

$$T(z, y) = (1+z^2y)/(1-z-z^2y). \tag{10}$$

Then

$$T(z, \alpha^3x) = (1+z^2\alpha^3x)/(1-z-z^2\alpha^3x); \tag{11}$$

$$T(z, \beta^3x) = (1+z^2\beta^3x)/(1-z-z^2\beta^3x). \tag{12}$$

We now note that $(1-z\alpha^2)(1+z\alpha x) = 1+z\alpha(x-\alpha)-z^2\alpha^3x = 1+z\alpha\beta-z^2\alpha^3x = 1-z-z^2\alpha^3x$. Similarly, we find that $(1-z\beta^2)(1+z\beta x) = 1-z-z^2\beta^3x$. We may also verify the following:

$$T(z, \alpha^3z) = -1+(1-z\alpha^2)^{-1}+(1+z\alpha x)^{-1}; \tag{13}$$

$$T(z, \beta^3z) = -1+(1-z\beta^2)^{-1}+(1+z\beta x)^{-1}. \tag{14}$$

Then, by expansion in (13) and (14):

$$T(z, \alpha^3x) = 1+\sum_{n=1}^{\infty}(\alpha^{2n}+(-x)^n\alpha^n)z^n;$$

$$T(z, \beta^3x) = 1+\sum_{n=1}^{\infty}(\beta^{2n}+(-x)^n\beta^n)z^n.$$

Now, using (9), we see that $R(z, x) = \theta^{-1}\sum_{n=0}^{\infty} z^n\{\alpha^{2n}-\beta^{2n}+(-x)^n(\alpha^n-\beta^n)\}$, or

$$R(z, x) = \sum_{n=0}^{\infty} z^n\{F_{2n}+(-x)^nF_n\}. \tag{15}$$

Likewise, $S(z, x) = 2+\sum_{n=1}^{\infty} z^n\{\alpha^{2n}-\beta^{2n}+(-x)^n(\alpha^n+\beta^n)\}$, or

$$S(z, x) = 2+\sum_{n=1}^{\infty} z^n\{L_{2n}+(-x)^nL_n\}. \tag{16}$$

Comparing the coefficients of z^n in (15) and (16) with those in (7) yields the desired results:

$$U_n(x) = F_{2n}+(-x)^nF_n,\ V_n(x) = L_{2n}+(-x)^nL_n,\ n = 1, 2, \ldots \text{ Q.E.D.} \tag{17}$$

Note: The Fibonacci polynomials are defined provided $x^2+4\neq 0$, i.e., $x\neq\pm 2i$. However, we may extend the definition of these polynomials to such exceptional values using continuity, i.e., by defining $F_n(2i) = ni^{n-1}$, $F_n(-2i) = n(-i)^{n-1}$. We also obtain $L_n(2i) = 2i^n$, $L_n(-2i) = 2(-i)^n$. With such definitions, we find that the results of the problem are indeed true for all complex x, including these exceptional values.

Also solved by A. J. Stam and the proposer.

❖❖❖