# APPLICATION OF THE ε-ALGORITHM TO THE RATIOS OF *r*-GENERALIZED FIBONACCI SEQUENCES

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## 1. INTRODUCTION

Let $a_0, a_1, \dots, a_{r-1}$ $(r \geq 2)$ be some real or complex numbers with $a_{r-1} \neq 0$. An $r$-generalized Fibonacci sequence $\{V_n\}_{n\geq 0}$ is defined by the linear recurrence relation of order $r$,

$$V_{n+1} = a_0 V_n + a_1 V_{n-1} + \cdots + a_{r-1} V_{n-r+1} \quad \text{for } n \geq r-1, \tag{1}$$

where $V_0, V_1, \dots, V_{r-1}$ are specified by the initial conditions. Such sequences are widely studied in the literature (see, e.g., [5], [6], [9], [10], [11], and [13]). We shall refer to them in the sequel as *sequences (1)*. It is well known that, if the limit $q = \lim_{n\to+\infty} \frac{V_{n+1}}{V_n}$ exists, then $q$ is a root of the characteristic equation $x^r = a_0 x^{r-1} + \cdots + a_{r-2} x + a_{r-1}$. Hence, sequences (1) may also be used as a tool in the approximation of roots of algebraic equations (see [12]), like Newton's method or the secant method as it was considered in [7].

The Aitken acceleration $\{x_n^*\}_{n\geq 0}$ associated with a convergent sequence $\{x_n\}_{n\geq 0}$ is defined by

$$x_n^* = \frac{x_{n+1}x_n - x_n^2}{x_{n+1} - 2x_n + x_{n-1}}. \tag{2}$$

For numerical analysis, this process is of practical interest in those cases in which $\{x_n^*\}_{n\geq 0}$ converges faster than $\{x_n\}_{n\geq 0}$ to the same limit (see, e.g., [1], [2], [3], [4], and [8]). In the case of sequences (1) with $r = 2$, McCabe and Philips had considered a theoretical application of Aitken acceleration for the accelerability of convergence of $\{x_n\}_{n\geq 0}$, where $x_n = \frac{V_{n+1}}{V_n}$ (see [12]). This is nothing more than the application of Aitken acceleration to the solution of the quadratic equation $x^2 - a_0 x - a_1 = 0$ by an iterative method (see [12]).

The main purpose of this paper is to apply the method of the ε-algorithm (see [3], [4]), which generalizes the Aitken acceleration, to accelerate the convergence of $\{x_n\}_{n\geq 0}$, where $x_n = \frac{V_{n+1}}{V_n}$, for any sequence (1). Hence, we extend the idea of McCabe and Philips [12] to the general case of sequences (1). Thus, we get the acceleration of the solution of algebraic equations.

This paper is organized as follows. In Section 2 we give a preliminary connection between sequences (1) and the ε-algorithm. In Section 3 we apply the ε-algorithm to the sequence of the ratios $x_n = \frac{V_{n+1}}{V_n}$. Some concluding remarks are given in Section 4.

## 2. SEQUENCES (1) AND THE ε-ALGORITHM

Let $\{x_n\}_{n\geq 0}$ be a convergent sequence of real numbers with $x = \lim_{n\to+\infty} x_n$. The ε-algorithm is a particular case of the extrapolation method (see [2], [3], [4]). The main idea is to consider a sequence transformation $T$ of $\{x_n\}_{n\geq 0}$ into a sequence $\{T_n\}_{n\geq 0}$, which converges very quickly to the same limit $x$, this means that $\lim_{n\to+\infty}\frac{T_n-x}{x_n-x}=0$ (see [3] and [4] for more details). The kernel of the transformation $T$, defined by

$$\mathscr{K}_T=\{\{x_n\}_{n\geq 0};\ \exists N>0,\ T_n=x,\ \forall n>N\},$$

is of great interest for an extrapolation method like Richardson or ε-algorithm (see [3], [4]). In summary, the ε-algorithm associated with the convergent sequence $\{x_n\}_{n\geq 0}$ consists in considering the following sequence $\{\varepsilon_k^{(n)}\}_{k\geq -1,\, n\geq 0}$, where

$$\varepsilon_{-1}^{(n)}=0,\ \varepsilon_1^{(n)}=x_n;\ n\geq 0, \tag{3}$$

$$\varepsilon_{k+1}^{(n)}=\varepsilon_{k-1}^{(n)}+\frac{1}{\varepsilon_k^{(n+1)}-\varepsilon_k^{(n)}},\ n,k\geq 0. \tag{4}$$

This algorithm can be applied when $\varepsilon_k^{(n)}\neq\varepsilon_k^{(n+1)}$ for any $n$, $k$. The ε-algorithm theory also shows that the only interesting quantities are $\varepsilon_{2k}^{(n)}$, the quantities $\varepsilon_{2k+1}^{(n)}$ are used only for intermediate computations (see [2], [3], [4]). For $k=2$, we can derive from expressions (3) and (4) that $\varepsilon_2^{(n)}$ is nothing but the Aitken acceleration associated with $\{x_n\}_{n\geq 0}$ as defined by (2) (see [3], [4]).

For any convergent sequence $\{x_n\}_{n\geq 0}$ with $x=\lim_{n\to+\infty}x_n$, Theorem 35 of [3] and Theorem 2.18 of [4] show that there exists $N>0$ such that $\varepsilon_{2k}^{(n)}=x$ for any $n\geq N$ if and only if there exists $a_0,\dots,a_k$ with $\sum_{j=0}^k a_j\neq 0$ such that $\sum_{j=0}^k a_j(x_{n+j}-x)=0$ for any $n\geq N$. It is easy to see that we can suppose in the last preceding sum that $a_0\neq 0$ and $a_k\neq 0$. Hence, we derive the following property.

***Proposition 2.1:*** Let $\{x_n\}_{n\geq 0}$ be a convergent sequence such that $x=\lim_{n\to+\infty}x_n$. Then the following are equivalent:

***(a)*** There exists $N>0$ such that $\varepsilon_{2k}^{(n)}=x$ for any $n\geq N$.

***(b)*** The sequence $\{V_n\}_{n\geq 0}$ defined by $V_n=x_{n+N}-x$ is a sequence (1) corresponding to $r=k$, whose coefficients and initial conditions are, respectively,

$$b_0=-\frac{a_{k-1}}{a_k},\dots,b_{k-1}=-\frac{a_0}{a_k}\quad\text{and}\quad V_0=x_N-x,\dots,V_{k-1}=x_{N+k-1}-x.$$

***(c)*** The sequence $\{x_n\}_{n\geq N}$ is a sequence (1) corresponding to $r=k+1$ such that $\lambda=1$ is a simple characteristic root, $V_0=x_N,\dots,V_k=x_{N+k}$ are its conditions, and its coefficients $a_0,\dots,a_k$ are the coefficients of the characteristic polynomial $P(X)=(X-1)Q(X)$, where $Q(X)$ is the characteristic polynomial of $\{V_n\}_{n\geq 0}$ defined in (b).

Proposition 2.1 shows that, in the case of the ε-algorithm, the kernel $\mathscr{K}_T$ may be expressed using sequences (1).

## 3. APPLICATION OF THE ε-ALGORITHM TO $\lim_{n\to+\infty}\frac{V_{n+1}}{V_n}$

Let $\{V_n\}_{n\geq 0}$ be a sequence (1) and $\lambda_0,\dots,\lambda_l$ be the roots of the characteristic polynomial $P(X)=X^r-a_0X^{r-1}-\cdots-a_{r-1}$. Suppose that $\lambda_0$ is a simple root and

$$0<|\lambda_l|\leq|\lambda_{l-1}|\leq\cdots\leq|\lambda_1|<|\lambda_0|.$$

Thus, the Binet formula of the sequence (1) is

$$V_n = \beta_{00}\lambda_0^n + \sum_{s=1}^{l}\left(\sum_{j=0}^{s_j-1}\beta_{js}n^n\right)\lambda_s^n,$$

where the $\beta_{js}$ are given by the initial conditions and $s_j$ is the multiplicity of $\lambda_j$ $(0 \le j \le l)$ (see, e.g., [9] and [10]). Suppose that $V_0, \dots, V_{r-1}$ are such that $\beta_{00} \neq 0$. Then we can derive that $\lim_{n\to+\infty}\frac{V_{n+1}}{V_n} = \lambda_0$.

It is known that if we applied the Aitken acceleration process to a convergent sequence $\{x_n\}_{n\ge0}$ with $x = \lim_{n\to+\infty} x_n$ and if $\lim_{n\to+\infty}\frac{x_{n+1}-x}{x_n-x} = \rho \neq 1$, then the sequence $\{\varepsilon_2^{(n)}\}_{n\ge0}$ converges more quickly than $\{x_n\}_{n\ge0}$ to $x$ (see [3], Theorem 32, p. 37). In the case of $x_n = \frac{V_{n+1}}{V_n}$, a direct computation using the Binet formula results in

$$\lim_{n\to+\infty}\frac{x_{n+1}-\lambda_0}{x_n-\lambda_0} = \frac{\lambda_1}{\lambda_0} \neq 1$$

because $|\lambda_1| < |\lambda_0|$. Hence, we have derived the following property.

***Proposition 3.1:*** Let $\{V_n\}_{n\ge0}$ be a sequence (1). Suppose that the characteristic roots $\{\lambda_j\}_{j=0}^{l}$ are such that $0 < |\lambda_l| \le |\lambda_{l-1}| \le \cdots \le |\lambda_1| < |\lambda_0|$ with $\lambda_0$ simple. Apply the Aitken acceleration to

$$\left\{x_n = \frac{V_{n+1}}{V_n}\right\}_{n\ge0}.$$

Then, the sequence $\{\varepsilon_2^{(n)}\}_{n\ge0}$ converges faster than $\{x_n\}_{n\ge0}$ to $\lambda_0$.

Let $\{x_n\}_{n\ge0}$ be a convergent sequence with $x = \lim_{n\to+\infty} x_n$. If $x_n = f(x_{n-1}, \dots, x_{n-k})$, where $x_0, \dots, x_{k-1}$ are given and

$$\sum_{i=0}^{r-1}\frac{\partial f}{\partial y_i}(x, \dots, x) \neq 1,$$

then $\lim_{n\to+\infty}\varepsilon_{2k}^{(n)} = x$ (see [3], Theorem 52, p. 70). Let $f$ be the function $f : \mathbf{D} \subset \mathbf{R}^{r-1} \to \mathbf{R}$, where $\mathbf{D} = \{(y_1, \dots, y_{r-1}) \in \mathbf{R}^{r-1};\ y_j \neq 0, \forall j\ (1 \le j \le r-1)\}$, defined by

$$f(y_1, \dots, y_{r-1}) = a_0 + \frac{a_1}{y_1} + \frac{a_2}{y_1y_2} + \cdots + \frac{a_{r-1}}{y_1\cdots y_{r-1}}.$$

Consider the ratio $x_n = \frac{V_{n+1}}{V_n}$. Then, from expression (1), we derive that $x_n = f(x_{n-1}, \dots, x_{n-r+1})$. It is clear that $f$ is a class $C^1$ on $\mathbf{D}$. By direct computation we obtain

$$\sum_{i=1}^{r-1}\frac{\partial f}{\partial y_i}(\lambda_0, \dots, \lambda_0) = 1 - \frac{1}{\lambda_0^{r-1}}\frac{dP}{dx}(\lambda_0).$$

Then we have derived the following result.

***Proposition 3.2:*** Let $\{V_n\}_{n\ge0}$ be a sequence (1). Suppose that the characteristic roots $\{\lambda_j\}_{j=0}^{l}$ are such that $\lambda_0$ is simple and $0 < |\lambda_l| \le |\lambda_{l-1}| \le \cdots \le |\lambda_1| < |\lambda_0|$. Apply the ε-algorithm to the sequence $\left\{x_n = \frac{V_{n+1}}{V_n}\right\}_{n\ge0}$. Then we have $\lim_{n\to+\infty}\varepsilon_{2(r-1)}^{(n)} = \lambda_0$.

More precisely, we have the following result.

***Proposition 3.3:*** Let $\{V_n\}_{n\geq 0}$ be a sequence (1). Suppose that the characteristic roots $\{\lambda_j\}_{j=0}^{l}$ are such that $\lambda_0$ is simple and $0<|\lambda_l|\leq|\lambda_{l-1}|\leq\cdots\leq|\lambda_1|<|\lambda_0|$. Apply the ε-algorithm to the sequence $\left\{x_n=\frac{V_{n+1}}{V_n}\right\}_{n\geq 0}$. Then the sequence $\{\varepsilon_{2(r-1)}^{(n)}\}_{n\geq 0}$ converges faster than $\{x_{n+r-1}\}_{n\geq 0}$ to $\lambda_0$.

***Proof:*** Let $b_j=\frac{\partial f}{\partial y_j}(\lambda_0,\ldots,\lambda_0)$. Then there exists $b_j^{(n)}$ $(1\leq j\leq r-1)$ such that

$$(\varepsilon_{2(r-1)}^{(n)}-\lambda_0)\left(-1+\sum_{j=1}^{r-1}b_j^{(n)}\right)=\sum_{j=1}^{r-1}(b_j^{(n)}-b_j)(x_{n-j}-\lambda_0)-R_n, \tag{$*$}$$

where

$$R_n=(x_n-\lambda_0)-b_1(x_{n-1}-\lambda_0)-\cdots-b_{r-1}(x_{n-r+1}-\lambda_0). \tag{$**$}$$

The application $(x_{n-r+1},\ldots,x_{n+r-1})\to(b_1^{(n)},\ldots,b_{r-1}^{(n)})$ is continuous (see [3] and [4]). Hence, for any $\varepsilon>0$, there exists $N>0$ such that $|b_j^{(n)}-b_j|<\varepsilon$ for any $n>N$ with $j=1,\ldots,r-1$. Then, from $(*)$, we derive that

$$\lim_{n\to+\infty}\frac{\varepsilon_{2(r-1)}^{(n)}-\lambda_0}{x_{n+r-1}-\lambda_0}=\frac{1}{-1+\sum_{j=1}^{r-1}b_j}\lim_{n\to+\infty}\frac{R_n}{x_{n+r-1}-\lambda_0}.$$

From expression $(**)$ of $R_n$, we obtain that

$$\lim_{n\to+\infty}\frac{R_n}{x_{n+r-1}-\lambda_0}=\left(\frac{\lambda_0}{\lambda_1}\right)^r-b_1\left(\frac{\lambda_0}{\lambda_1}\right)^{r+1}-\cdots-b_{r-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{2r-2}.$$

A direct computation using the expression

$$b_j=-\frac{a_j}{\lambda_0^{j+1}}-\frac{a_{j+1}}{\lambda_0^{j+2}}-\cdots-\frac{a_{r-1}}{\lambda_0^{r}}\quad(1\leq j\leq r-1),$$

results in $\lim_{n\to+\infty}\frac{R_n}{x_{n+r-1}-\lambda_0}=0$. Thus, we have

$$\lim_{n\to+\infty}\frac{\varepsilon_{2(r-1)}^{(n)}-\lambda_0}{x_{n+r-1}-\lambda_0}=0.\ \square$$

The proof of Proposition 3.3 is nothing more than an adaptation of the proof of Theorem 52 of [3] to the case in which

$$f(y_1,\ldots,y_r)=a_0+\frac{a_1}{y_1}+\frac{a_2}{y_1y_2}+\cdots+\frac{a_{r-1}}{y_1\cdots y_{r-1}}.$$

## 4. CONCLUDING REMARKS

Note that the ε-algorithm may also be used to accelerate the convergence of sequences (1). More precisely, for a convergent sequence (1), the Binet formula results in $|\lambda_j|\leq 1$ for any characteristic root $\lambda_j$ $(0\leq j\leq l)$. Suppose that $0<|\lambda_1|<\cdots<|\lambda_1|<|\lambda_0|\leq 1$. Then the Binet formula and expression (1) imply that $\lim_{n\to+\infty}V_n=0$ for $|\lambda_j|<1$ for any $j$ or $\lim_{n\to+\infty}V_n=\beta_{00}$ if $|\lambda_j|<1$ for any $j\neq 0$, and $\lambda_0=1$ is a simple characteristic root.

For $\lim_{n\to+\infty} V_n = 0$, we show by direct computation that $\lim_{n\to+\infty}\frac{V_{n+1}}{V_n} = \lambda_j$, depending on the choice of the initial conditions $\{V_k\}_{k=0}^{r-1}$. Then, by applying the ε-algorithm, we can derive that $\{\varepsilon_{2p}^{(n)}\}_{n\geq 0}$ converges to 0 faster than $\{V_n\}_{n\geq 0}$, for any $p = 1, \ldots, r-j$.

For $\lim_{n\to+\infty} V_n = \beta_{00} = S \neq 0$, we can derive by direct computation that $\lim_{n\to+\infty}\frac{V_{n+1}-S}{V_n-S} = \lambda_j$, depending on the choice of the initial conditions $\{V_k\}_{k=0}^{r-1}$. Then, by applying the ε-algorithm, we also derive that $\{\varepsilon_{2p}^{(n)}\}_{n\geq 0}$ converges to $S$ faster than $\{V_n\}_{n\geq 0}$ for any $p = 1, \ldots, r-j$. In particular, this case may be used to accelerate the convergence of the ratios $\frac{V_n}{q^n}$ when the $a_j$ are nonnegative and $\mathrm{CGD}\{j+1; a_j > 0\} = 1$ (see [6] and [14]).

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