ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by

Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2001. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-911 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***

Determine whether $L_n + 2(-1)^m L_{n-2m-1}$ is divisible by 5 for all positive integers m and n.

B-912 ***Proposed by the editor***

Express $F_{n+(n \bmod 2)} \cdot L_{n+1-(n \bmod 2)}$ as a sum of Fibonacci numbers.

B-913 ***Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA***

Fix an integer $k \geq 1$. The Fibonacci numbers satisfy an "accelerated" recurrence of the form

$$F_{n2^k} = \alpha_k F_{(n-1)2^k} - F_{(n-2)2^k} \quad (n = 2, 3, \ldots)$$

with $F_0 = 0$ and F_{2^k} to start the recurrence. For example, when $k = 1$, we have

$$F_{2n} = 3F_{2(n-1)} - F_{2(n-2)} \quad (n = 2, 3, \ldots; F_0 = 0; F_2 = 1).$$

a. Find the constant α_k by identifying it as a certain member of a sequence that is known by readers of these pages.

b. Generalize this result by similarly identifying the constant β_m for which the accelerated recurrence

$$F_{mn+h} = \beta_m F_{m(n-1)+h} + (-1)^{m+1} F_{m(n-2)+h},$$

with appropriate initial conditions, holds.

B-914 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain***

Let $n \geq 2$ be an integer. Prove that

$$\prod_{k=2}^{n}\left\{\sum_{j=1}^{k}\frac{1}{(F_{k+2}-F_j-1)^2}\right\} \geq \frac{1}{F_2F_{n+1}}\left(\frac{n}{F_3F_4\ldots F_n}\right)^2.$$

B-915 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN***

If $|x| \leq 1$, prove that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_jx^{i-1}\right| < n^3.$$

SOLUTIONS

An Exponential Equation with Fibonacci Base

B-894 ***Proposed by the editor***
(Vol. 38, no. 1, February 2000)

Solve for x: $F_{110}^x + 442F_{115}^x + 13F_{119}^x = 221F_{114}^x + 255F_{117}^x$.

Solution by Paul S. Bruckman, Berkeley CA

The desired value of x must clearly be rational, if it exists at all. We may deduce the appropriate value of x by an approximation technique. If we replace F_{110} by u, say, then it is *approximately* true that $F_{114} = u\alpha^4$, $F_{115} = u\alpha^5$, $F_{117} = u\alpha^7$, and $F_{119} = u\alpha^9$. Therefore, the desired equation is replaced by the following approximate equation:

$$13\alpha^{9x} - 255\alpha^{7x} + 442\alpha^{5x} - 221\alpha^{4x} + 1 \approx 0. \tag{1}$$

Let $G(x)$ represent the expression in the left member of (1). It is easily found that $G(0) = -20$. We may also compute the following approximate values: $G(1) \approx -3027.513$, $G(2) \approx -95869.589$, $G(3) = 0$ (exactly), $G(4) \approx 2.58992{*}10^8$. Clearly, $G(x)$ increases without bound for all $x \geq 4$, since the term $13\alpha^{9x}$ dominates $G(x)$. Therefore, it appears that $x = 3$ is the unique desired solution; however, this must be verified in the exact (original) equation.

The cubes of the Fibonacci numbers have a characteristic polynomial of degree 4; if $P_3(z)$ is this polynomial, it is easily seen that

$$P_3(x) = (z-\alpha^3)(z-\alpha^2\beta)(z-\alpha\beta^2)(z-\beta^3) = (z^2-4z-1)(z^2+z-1)$$
$$= (z^2-1)^2 - 3z(z^2-1) - 4z^2 = z^4 - 6z^2 + 1 - 3z^3 + 3z$$

or

$$P_3(z) = z^4 - 3z^3 - 6z^2 + 3z + 1. \tag{2}$$

That is, we have the following recurrence relation for the cubes of the Fibonacci numbers, valid for all n:

$$(F_{n+4})^3 - 3(F_{n+3})^3 - 6(F_{n+2})^3 + 3(F_{n+1})^3 + (F_n)^3 = 0. \tag{3}$$

Using (3), we need to verify the following relation:

$$13(F_{119})^3 - 255(F_{117})^3 + 442(F_{115})^3 - 221(F_{114})^3 + (F_{110})^3 = 0. \tag{4}$$

From (3), we gather that

$$(F_{110})^3 = -(F_{114})^3 + 3(F_{113})^3 + 6(F_{112})^3 - 3(F_{111})^3.$$

Also,

$$(F_{119})^3 = 3(F_{118})^3 + 6(F_{117})^3 - 3(F_{116})^3 - (F_{115})^3;$$

thus, if S represents the expression in the left member of (4), we obtain (after some simplification):

$$S = 39(F_{118})^3 - 177(F_{117})^3 - 39(F_{116})^3 + 429(F_{115})^3 - 222(F_{114})^3 + 3(F_{113})^3 + 6(F_{112})^3 - 3(F_{111})^3.$$

Also,

$$(F_{111})^3 = -(F_{115})^3 + 3(F_{114})^3 + 6(F_{113})^3 - 3(F_{112})^3$$

and

$$(F_{118})^3 = 3(F_{117})^3 + 6(F_{116})^3 - 3(F_{115})^3 - (F_{114})^3.$$

After further simplification, we obtain:

$$S = -60(F_{117})^3 + 195(F_{116})^3 + 315(F_{115})^3 - 270(F_{114})^3 - 15(F_{113})^3 + 15(F_{112})^3.$$

Finally, we make the substitutions:

$$(F_{112})^3 = -(F_{116})^3 + 3(F_{115})^3 + 6(F_{114})^3 - 3(F_{113})^3$$

and

$$(F_{117})^3 = 3(F_{116})^3 + 6(F_{115})^3 - 3(F_{114})^3 - (F_{115})^3.$$

Then, after further simplification, we obtain $S = 0$, identically. This establishes (4) and shows that $x = 3$ is the unique solution to the problem.

Brian D. Beasley noted that the solution $x = 3$ works for n, $n+5$, $n+9$, $n+4$, $n+7$ in place of 110, 115, 119, 114, and 117, respectively.

Also solved by Brian D. Beasley, Indulis Strazdins, and the proposer.

A Recurrence for F_{n^2}

B-895 ***Proposed by Indulis Strazdins, Riga Technical University, Latvia (Vol. 38, no. 2, May 2000)***

Find a recurrence for F_{n^2}.

Solution by Paul S. Bruckman, Berkeley CA

For brevity, write $Q_n \equiv F_{n^2}$. We may easily demonstrate that the following recurrence relation is satisfied:

$$Q_{n+1} - Q_{n-1} = F_{2n}L_{n^2+1}. \tag{1}$$

We may verify (1), using the identity,

$$F_uL_v = F_{v+u} - (-1)^u F_{v-u}, \tag{2}$$

by setting $u = 2n$, $v = n^2 + 1$. Also, setting $u = 4n$, $v = n^2 + 4$ yields

$$Q_{n+2} - Q_{n-2} = F_{4n}L_{n^2+4}. \tag{3}$$

Now we note (or easily verify) the following identities:

$$L_{m+4} = 7L_{m+1} - 10F_m \tag{4}$$

and

$$F_{4n} = F_{2n}L_{2n}. \tag{5}$$

Then, setting $m = n^2$ in (4) we obtain, from (1) and (3):

$$Q_{n+2} - Q_{n-2} - 7L_{2n}(Q_{n+1} - Q_{n-1}) + 10F_{4n}Q_n = 0. \tag{6}$$

Since $Q_{-n} = Q_n$, we see that (6) is valid for all integral n. We may also express (6) in the asymmetric form:

$$Q_{n+4} = 7L_{2n+4}Q_{n+3} - 10F_{4n+8}Q_{n+2} - 7L_{2n+4}Q_{n+1} + Q_n. \tag{7}$$

It does not appear that we can obtain a linear recurrence for the Q_n's that contains constant coefficients.

H.-J. Seiffert gave the formula

$$G_{n+2} = \frac{(F_{2n+1}L_{2n+3} - 1)G_{n+1} + F_{2n+3}G_n}{F_{n_{2n+1}}},$$

where $G_n = F_{n^2}$, *and L. A. G. Dresel gave the formula*

$$T_{n+1} = \frac{1}{2}(5S_n + T_n)L_{2n} - T_{n-1},$$

where $S_n = F_{n^2}$ *and* $T_n = L_{n^2}$. *He also noted that the factor* L_{2n} *occurring in the above formula can be obtained from the formula* $L_{2(n+1)} = 3L_{2n} - L_{2(n-1)}$.

Also solved by L. A. G. Dresel, H.-J. Seiffert, and the proposer.

An Independent Constant Fibonacci Sum

B-896 ***Proposed by Andrew Cusumano, Great Neck, NY***
(Vol. 38, no. 2, May 2000)

Find an integer k such that the expression $F_n^4 + 2F_n^3F_{n+1} + kF_n^2F_{n+1}^2 - 2F_nF_{n+1}^3 + F_{n+1}^4$ is a constant independent of n.

Solution by Kee-Wai Lau, Hong Kong, China

We show that, for $k = 1$, the given expression equals 1. This amounts to proving that

$$F_n^4 + 2F_n^3F_{n+1} - F_n^2F_{n+1}^2 - 2F_nF_{n+1}^3 + F_{n+1}^4 - 1 = 0. \tag{1}$$

In fact, the left-hand side of (1) equals

$$\begin{aligned} &F_n(F_n + F_{n+1})(F_n - F_{n+1})(F_{n+1} + (F_{n+1} + F_n)) + (F_{n+1}^2 + 1)(F_{n+1}^2 - 1) \\ &= -F_nF_{n+2}F_{n-1}F_{n+3} + (F_{n+1}^2 + 1)(F_{n+1}^2 - 1). \end{aligned}$$

Hence, to prove (1), it suffices to show that

$$F_{n+1}^2 - F_nF_{n+2} = (-1)^n \tag{2}$$

and

$$F_{n+1}^2 - F_{n-1}F_{n+3} = (-1)^{n-1}. \tag{3}$$

However, both (2) and (3) can be established readily by using the relation

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \text{ where } \alpha > 0,\ \alpha + \beta = 1, \text{ and } \alpha\beta = -1.$$

This completes the solution.

R. J. Hendel noted that the result follows from the "verification theorem" in L. A. G. Dresel's article "Transformation of Fibonacci-Lucas Identities" in Application of Fibonacci Numbers **5**, *ed. G. E. Bergum et al. (Dordrecht: Kluwer, 1993), after one trivially verifies that, for* $k = -1$, *the expression evaluates to 1 for* $n = -2$ *to* $n = 2$.

Also solved by P. Bruckman, C. Cook, L. A. G. Dresel, R. J. Hendel, H. Kwong, K. Lewis, H.-J. Seiffert, and J. Sellers.

An Initial Value Problem

B-897 ***Proposed by Brian D. Beasley, Presbyterian College, Clinton, SC (Vol. 38, no. 2, May 2000)***

Define $\langle a_n \rangle$ by $a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$ with initial conditions $a_0 = 4$, $a_1 = 2$, and $a_2 = 10$. Express a_n in terms of Fibonacci and/or Lucas numbers.

Solution by Richard André-Jeannin, Cosnes et Romain, France

The characteristic polynomial of the proposed recurrence can be easily factorized:

$$X^3 - 2X^2 - 2X + 1 = (X - \alpha^2)(X - \beta^2)(X + 1).$$

From this, we see that there exist constants A, B, and C such that $a_n = A\alpha^{2n} + B\beta^{2n} + C(-1)^n$. Considering the initial conditions, we obtain the linear system:

$$\begin{cases} A + B + C = 4, \\ A\alpha^2 + B\beta^2 - C = 2, \\ A\alpha^4 + B\beta^4 + C = 10. \end{cases}$$

After some calculations, we get $A = B = \frac{6}{5}$ and $C = \frac{8}{5}$; thus,

$$\begin{aligned} a_n &= \frac{6(\alpha^{2n} + \beta^{2n}) + 8(-1)^n}{5} = \frac{6L_{2n} + 8(-1)^n}{5} \\ &= L_{2n} + 2(-1)^n + \frac{L_{2n} - 2(-1)^n}{5} = L_n^2 + F_n^2. \end{aligned}$$

Also solved by P. Bruckman, J. Cigler, C. Cook, K. Davenport, L. A. G. Dresel, H. Kwong, K. Lewis, D. Redmond, M. Rose, H.-J. Seiffert, J. Sellers, I. Strazdins, and the proposer.

Some Fibonacci Sum

B-898 ***Proposed by Alexandru Lupaş, Sibiu, Romania (Vol. 38, no. 2, May 2000)***

Evaluate

$$\sum_{k=0}^{s} (-1)^{(n-1)(s-k)} \binom{2s+1}{s-k} F_{n(2k+1)}.$$

Solution by L. A. G. Dresel, Reading, England

Denoting the given expression by $E(s,n)$, consider the $2s+2$ terms of the expansion of $(\alpha^n-\beta^n)^{2s+1}$. The two central terms combine to give

$$\binom{2s+1}{s}(-\alpha^n\beta^n)^s(\alpha^n-\beta^n)=\binom{2s+1}{s}(-1)^{(n-1)s}(\sqrt{5})F_n.$$

Then proceeding outward and combining pairs of terms equidistant from the center, we obtain $(\alpha^n-\beta^n)^{2s+1}=(\sqrt{5})E(s,n)$. Therefore, since $(\alpha^n-\beta^n)=(\sqrt{5})F_n$, we have $E(s,n)=5^s(F_n)^{2s+1}$.

Note: The result corresponds to equation (80) in *Fibonacci & Lucas Numbers, and the Golden Section,* by S. Vajda (Chichester: Ellis Horwood Ltd., 1989).

Don Redmond obtained a similar result for a Lucas analog to the problem. He showed that

$$\sum_{k=0}^{s}(-1)^{n(s-k)}\binom{2s+1}{s-k}L_{n(2k+1)}=L_n^{2s+1}.$$

Also solved by P. Bruckman, J. Cigler, D. Redmond, H.-J. Seiffert, I. Strazdins, and the proposer.

On a Bruckman Conjecture

A Comment by N. Gauthier, Canada

In the Feb. 2000 issue of this quarterly, Dr. Rabinowitz published the solution to Elementary Problem B-871 by Paul S. Bruckman ["Absolute Sums," *The Fibonacci Quarterly* **38.1** (2000):86-87]. In a footnote to Indulis Strazdin's solution, Dr. Rabinowitz then commented that "*Bruckman noted that*

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|=n\binom{2n}{n}$$

and conjectured that

$$\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^{2r-1}=P_r(n)\binom{2n}{n}$$

for some monic polynomial $P_r(n)$ of degree r." Here, $n\geq 0$ and $r\geq 1$ are integers.

Professor Bruckman's conjecture about the general form of the latter sum seems to be correct based on evidence I collected with MAPLE V(R), but contrary to the conjecture, $P_r(n)$ is generally not a monic polynomial in n. From the evidence collected, the leading coefficient of $P_r(n)$ appears to be equal to $(r-1)!$; for $r=1$ and $r=2$, one then finds the leading terms to be n and n^2, respectively, in agreement with the values presented in the solution of problem B-871. But, for $r\geq 3$, the leading term of $P_r(n)$ is $(r-1)!n^r$.

The Bruckman polynonmials $P_r(n)$ were obtained for $1\leq r\leq 20$, using MAPLE V(R), by noting that

$$P_r(n)=\frac{1}{\binom{2n}{n}}\sum_{k=0}^{2n}\binom{2n}{k}|n-k|^{2r-1}=\frac{2}{\binom{2n}{n}}\sum_{k=0}^{n-1}\binom{2n}{k}(n-k)^{2r-1}$$

and by asking MAPLE for $P_r(n)$. For easy reference, here are the values of $P_r(n)$ for $r=3$, 4, 5, and 6:

$P_3(n)=2n^3-n^2$; $\quad P_4(n)=6n^4-8n^3+3n^2$;

$P_5(n)=24n^5-60n^4+54n^3-17n^2$; $\quad P_6(n)=120n^6-480n^5+762n^4-556n^3+155n^2$.

(I will gladly provide the polynomials for $7\leq r\leq 20$ on request for interested readers who might not have access to MAPLE.)