ON THE SOLVABILITY OF A FAMILY OF DIOPHANTINE EQUATIONS

M. A. Nyblom and B. G. Sloss
Department of Mathematics, Royal Melbourne Institute of Technology,
GPO Box 2476V, Melbourne, Victoria 3001, Australia
(Submitted January 1999)

1. INTRODUCTION

A frequently occurring problem in the theory of binary quadratic forms is to determine, for a given integer m, the existence of solutions to the Diophantine equation

$$f(x,y):=ax^2+bxy+cy^2=m,$$

having discriminant $\Delta=b^2-4ac$. In the case of a strictly positive nonsquare discriminant, it is well known that the occurrence of one solution to $f(x,y)=m$ implies the existence of infinitely many other solutions. Using this fact, one may attempt to investigate the solvability of the following family of binomial Diophantine equations,

$$\binom{x+1}{2}=d\binom{y+1}{2}, \tag{1}$$

where $d\in N\setminus\{0\}$, as they can be recast into a quadratic form by completion of the square to obtain the following Pell-like equation

$$X^2-dY^2=1-d, \tag{2}$$

where $X=2x+1$ and $Y=2y+1$. Indeed, as (1, 1) is a solution of (2), there will exist infinitely many solutions (X,Y) when $\Delta=4d>0$ and is nonsquare. Unfortunately, in order to relate this to the solvability of our family of Diophantine equations, we must demonstrate that within the solution set of (2) there exists an infinite subset of solutions (X,Y) for which both X and Y are odd integers. To address such a problem, it will be necessary here to exploit a group action on the solution set $\mathcal{S}:=\{(x,y)\in Z^2: f(x,y)=m\}$, which allows one to generate an infinite subset of elements in $\mathcal{S}$ from a given solution in $\mathcal{S}$. In the case of (2) for a nonsquare $d\in N\setminus\{0\}$, an infinite subset of odd solutions can be generated from (1, 1). Although the solvability of (1) has been proved using elementary arguments (see [1]), the approach taken here is more direct and can be applied to a wider class of Diophantine equations. To illustrate, the above method will be used to establish, for each $m\in Z\setminus\{0\}$, the existence of infinitely many integer solutions to the more general family of equations

$$x(x+m)=dy(y+m), \tag{3}$$

when $d\in N\setminus\{0\}$ is nonsquare. The subset of solutions generated from the above group action are often referred to within the literature as orbits since they are closed with respect to the group action. It is well known (see [2]) that the solution set $\mathcal{S}$, when nonempty, is equal to a finite union of distinct orbits, each generated from a unique solution in $\mathcal{S}$. Consequently, in addition to proving the solvability of (3), we shall derive an asymptotic formula for the maximum number of distinct orbits that are required to completely describe $\mathcal{S}$ in the case of (3) as $d\to\infty$ through nonsquare values. Despite the reliance in this paper on algebraic methods, it is possible to demonstrate the solvability of the original class of Diophantine equations, for $d = 2, 3$, by a more elementary argument than that used in [1]. This method, which has already been applied to the case $d = 2$ in connection with the study of Pythagorean triples (see [3]), will result here in an algorithm for generating all positive integer solutions for the case $d = 3$. As an interesting aside, we further provide what the authors believe to be an unknown characterization for the solutions of the "negative Pell equation" $X^2 - 2Y^2 = -1$ in terms of the set of square triangular numbers; this follows as a direct consequence of the analysis in [3].

2. MAIN RESULT

We begin in this section by introducing some well-known concepts and results from the theory of binary quadratic forms that will be required in describing the group action on the set $\mathcal{S} = \{(x, y) \in Z^2 : f(x, y) = m\}$. The background material that follows has been taken from [2], where quadratic forms are treated from the perspective of quadratic number fields and their rings of integers. In what follows, assume Δ is a positive, nonsquare integer with $\Delta \equiv 0 \pmod 4$.

Definition 2.1: Let $Q(\sqrt{\Delta})$ be the quadratic extension of Q obtained by adjoining $\sqrt{\Delta}$. Define conjugation σ and norm N as follows: For $x, y \in Q$ and $\alpha = x + y\sqrt{\Delta}$, set $\sigma(\alpha) = x - y\sqrt{\Delta}$ and $N(\alpha) = \alpha\sigma(\alpha) = x^2 - \Delta y^2 \in Q$.

Using the well-known fact that $\sigma : Q(\sqrt{\Delta}) \to Q(\sqrt{\Delta})$ is an automorphism, it is easily deduced that the norm map N is multiplicative. In the theory of binary quadratic forms, the Pell equation plays a central role. We now introduce this equation and briefly examine the algebraic structure of its solution set.

Definition 2.2: The Pell equation is given by $f_\Delta(x, y) = 1$, where f_Δ is an integral binary form as follows:

$$f_\Delta(x, y) = x^2 - \frac{\Delta}{4}y^2,$$

with discriminant Δ. The negative Pell equation is $f_\Delta(x, y) = -1$. One also defines $Pell^{\pm}(\Delta) = \{(x, y) \in Z^2 : f_\Delta(x, y) = \pm 1\}$ and $Pell(\Delta) = \{(x, y) \in Z^2 : f_\Delta(x, y) = 1\}$.

For the above values of the discriminant Δ, it is known that $Pell(\Delta)$ has infinitely many elements. More importantly, all solutions with positive x and y can be generated as a power of a minimal "fundamental" solution. These results can be deduced by analyzing the Pell equation from the context of the subring $\mathcal{O}_\Delta$ of $Q(\sqrt{\Delta})$ having the underlying set $\{x + y\rho_\Delta : x, y \in Z\}$, where $\rho_\Delta = \sqrt{\Delta / 4}$. We expand here a little on this analysis, which not only leads to the group structure of $Pell(\Delta)$ but will also help to effect the desired group action of $\mathcal{S}$.

As every ordered pair $(x, y) \in Pell^{\pm}(\Delta)$ can be uniquely represented as an element $x + y\rho_\Delta \in \mathcal{O}_\Delta$, one sees from the calculation $N(x + y\rho_\Delta) = f_\Delta(x, y)$ that solving the positive or negative Pell equation is equivalent to finding the elements in $\mathcal{O}_\Delta$ having norm equal to ± 1. However, from the multiplicativity of N, it is easily established that $N(\alpha) = \pm 1$ for $\alpha \in \mathcal{O}_\Delta$ if and only if α is a unit in the ring $\mathcal{O}_\Delta$. Consequently, if one denotes the group of units in $\mathcal{O}_\Delta$ by $\mathcal{O}_\Delta^\times$, then $\psi := x + y\rho_\Delta$ defines a bijection $\psi : Pell^{\pm}(\Delta) \to \mathcal{O}_\Delta^\times$. Hence, $Pell^{\pm}(\Delta)$ is a group, as it is in bijection with the commutative group $\mathcal{O}_\Delta^\times$. Moreover, by using ψ to map the group law from $\mathcal{O}_\Delta^\times$, it is easily seen that the product of solutions in $Pell^\pm(\Delta)$ is given by

$$(u,v)\cdot(U,V)=\left(uU+\frac{\Delta}{4}vV, uV+vU\right). \tag{4}$$

If we further define $\mathcal{O}_{\Delta,1}^\times=\{\alpha\in\mathcal{O}_\Delta^\times : N(\alpha)=1\}$ and $\mathcal{O}_{\Delta,+}^\times=\{\alpha\in\mathcal{O}_\Delta^\times:\alpha>0\}$, then restricting ψ to the subgroup $\mathcal{O}_{\Delta,1}^\times$ of $\mathcal{O}_\Delta^\times$ gives an isomorphism $Pell(\Delta)\cong\mathcal{O}_{\Delta,1}^\times$. The cyclic nature of the group $Pell(\Delta)$ can be deduced by first noting that $\mathcal{O}_{\Delta,+}^\times$ contains a minimal element ε over all elements in $\mathcal{O}_{\Delta,+}^\times$ that are greater than unity (see [2]). Using this fact, it can be easily shown, as $\mathcal{O}_{\Delta,+}^\times\subseteq(0,\infty)=\bigcup_{n\in Z}[\varepsilon^n,\varepsilon^{n+1})$, that any $\alpha\in\mathcal{O}_{\Delta,+}^\times$ is of the form $\alpha=\varepsilon^n$ for some $n\in Z$. Since any $\beta=\pm\varepsilon^n$ for some $n\in Z$. Thus, if one formally defines

$$\tau_\Delta=\begin{cases}\varepsilon & \text{if } N(\varepsilon)=1,\\ \varepsilon^2 & \text{if } N(\varepsilon)=-1,\end{cases}$$

then $\mathcal{O}_{\Delta,1}^\times=\{\pm\tau_\Delta^m : m\in Z\}\cong Pell(\Delta)$.

Remark 2.1: Note that, if $\alpha=\mathcal{O}_\Delta^\times\backslash\mathcal{O}_{\Delta,+}^\times$, then as $-1=-1+0\rho_\Delta\in\mathcal{O}_\Delta^\times$ and $-\alpha\in\mathcal{O}_{\Delta,+}^\times$, one must have $-\alpha=\pm\varepsilon^n$ or $\alpha=\mp\varepsilon^n$ for some $n\in Z$. Consequently, if $\varepsilon=a+b\rho_\Delta$, then from the bijection $\psi: Pell^\pm(\Delta)\to\mathcal{O}_\Delta^\times$ it is clear that $Pell^\pm(\Delta)=\{\pm(x_n,y_n):n\in Z\}$, where $x_n+y_n\rho_\Delta=(a+b\rho_\Delta)^n$. Similarly, the solutions in $Pell(\Delta)=\{\pm(x_n,y_n):n\in Z\}$ can be calculated from $x_n+y_n\rho_\Delta=\tau_\Delta^n$. While in the case in which $N(\varepsilon)=-1$ the solutions in $Pell^-(\Delta)$ are obtained from the subset $\{\pm(x_{2n+1},y_{2n+1}):n\in Z\}$ of $Pell^\pm(\Delta)$ as they are the only ordered pairs in $Pell^\pm(\Delta)$ for which $N(\psi)=-1$.

To help define the group action on $\mathcal{S}$, we proceed in a similar manner to the above, by first generalizing the construction of the ring $\mathcal{O}_\Delta$. The definition given below is motivated by the factorization

$$f(x,y)=\frac{1}{a}\left(xa+y\frac{b+\sqrt{\Delta}}{2}\right)\left(xa+y\frac{b-\sqrt{\Delta}}{2}\right).$$

Definition 2.3: The module M_f of an integral binary quadratic form $f(x,y)$, which has discriminant Δ, is the $\mathcal{O}_\Delta$ module having the underlying set $\{xa+y(b+\sqrt{\Delta})/2 : x,y\in Z\}\subseteq Q(\sqrt{\Delta})$.

It is the closure of M_f under multiplication by elements in $\mathcal{O}_\Delta$ that most interests us here. The important calculation is $(u+v\rho_\Delta)(xa+y(b+\sqrt{\Delta})/2)=(x'a+y'(b+\sqrt{\Delta})/2)$, where

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}u-\frac{b}{2}v & -cv\\ av & u+\frac{b}{2}v\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}. \tag{5}$$

Equation (5) can be used to define an action of the group $\mathcal{O}_{\Delta,1}^\times$ on $\mathcal{S}$, which, given an $(x,y)\in\mathcal{S}$, one can generate an infinite set)or orbit) of solutions in $\mathcal{S}$ by repeated application of (5). To see this, first observe that any $(x,y)\in\mathcal{S}$ is uniquely represented as an element $ax+y(b+\sqrt{\Delta})/2\in M_f$. Now, as in the case of the Pell form, one can set $\psi(x,y):=xa+y(b+\sqrt{\Delta})/2$, from which it is immediate that $N(\psi(x,y))=af(x,y)$. Hence, ψ defines a bijection $\psi:\mathcal{S}\to\{\gamma\in M_f : N(\gamma)=am\}$. If we formally define the action of an element $\alpha\in\mathcal{O}_{\Delta,1}^\times$ on the set $\mathcal{S}$ by

$\alpha\cdot(x,y):=\psi^{-1}(\alpha\psi(x,y))$, then from the multiplicity of N it is clear that $N(\psi(\alpha\cdot(x,y)))=N(\alpha\psi(x,y))=am$, consequently, $\alpha\cdot(x,y)\in\mathcal{S}$. As $\mathbb{O}_{\Delta,1}^{\times}$ is a cyclic group of infinite order, the set $\mathcal{S}$, when nonempty, will at least contain the infinite subset of solutions in the orbit given by $\mathbb{O}_{\Delta,1}^{\times}\cdot(x,y)=\{\pm\tau_{\Delta}^{n}\cdot(x,y),n\in Z\}$. Also, from the bijection $\psi(x,y)$, the elements in $\mathbb{O}_{\Delta,1}^{\times}\cdot(x,y)$ can be calculated explicitly by repeated application of (5). We now apply the above group action to establish the solvability of the general Diophantine equation in (3).

Theorem 2.1: Suppose $d>1$ is a nonsquare integer and $m\in Z\setminus\{0\}$, then there exist infinitely many solutions to the Diophantine equation

$$x(x+m)=dy(y+m). \tag{6}$$

Proof: By completing the square, the Diophantine equation in question can be rewritten in the form

$$X^2-dY^2=m^2(1-d), \tag{7}$$

where $X=2x+m$ and $Y=2y+m$. When $m=2s$, equation (7) can be reduced further to the quadratic form

$$Z^2-dW^2=s^2(1-d), \tag{8}$$

where $Z=x+s$ and $W=y+s$. Now, for the assumed values of d, equation (8) has the nonsquare discriminant $\Delta=4d$ and so an infinite number of solutions can be generated from the orbit $\mathbb{O}_{\Delta,1}^{\times}\cdot(s,s)=\{(Z,W)=\pm\tau_{\Delta}^{n}\cdot(s,s):n\in Z\}$. Hence, the original Diophantine equation in (6) will have at least the infinite subset of solutions given by $\{(x,y)=(Z-s,W-s):(Z,W)\in\mathbb{O}_{\Delta,1}^{\times}\cdot(s,s)\}$. If m is odd, then the question of solvability of (6) is reduced to knowing whether there exist infinitely many odd solutions to (7). We now examine the orbit of solutions generated by the action of $\mathbb{O}_{\Delta,1}^{\times}$ on (m,m). If $\tau_{\Delta}=u+v\rho_{\Delta}$, then, by (5), the sequence of elements $\{\tau_{\Delta}^{n}\cdot(m,m)\}_{n=0}^{\infty}$ can be generated using

$$\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix}=\begin{pmatrix}u & dv\\ v & u\end{pmatrix}\begin{pmatrix}x_n\\ y_n\end{pmatrix} \tag{9}$$

with $(x_0,y_0)=(m,m)$. We claim that, for all nonsquare $d>1$, the sequence $\{(x_n,y_n)\}_{n=0}^{\infty}$ contains infinitely many odd ordered pairs. To demonstrate this by induction it will be convenient, since $u^2=1+dv^2$, to deal with the following cases separately. For brevity, one need only attend to the inductive step in each case.

Case 1. $2\nmid d$

In this instance, u and v will be of opposite parity. If, for some $n\geq 0$, it is assumed that (x_n,y_n) is an odd ordered pair, then by (9) both x_{n+1} and y_{n+1} are the sum of an odd and even number and so must be odd.

Case 2. $2\mid d$

Now u will always be odd irrespective of the parity of v. If v is even, then the oddness of the ordered pair (x_n,y_n) follows by an analogous argument to the one above. For v odd, we shall establish that all the odd solutions are contained in the subsequence $\{(x_{2n},y_{2n})\}_{n=0}^{\infty}$. Therefore, suppose x_{2n} and y_{2n} are odd for some $n\geq 0$, then by (9) x_{2n+1} is odd while y_{2n+1} is even.

However, by another application of (9), one finds that both x_{2n+2} and y_{2n+2} are the sum of an odd and an even integer and so must be odd. $\square$

Corollary 2.1: Suppose $d>1$ is a nonsquare integer and T_n denoted the n^{th} triangular number, then there exist infinitely many pairs of positive integers (m, n) such that $T_m = dT_n$. If d is a perfect square then, in general, only at most finitely many solutions (m, n) can be found while, in particular, non exist when $d = p^{2s}$ for p a prime.

Proof: The first statement follows from setting $m=1$ in Theorem 2.1. Suppose now d is a perfect square, with $m=\sqrt{d}$. Clearly, the equation $f(X,Y)=1-d$, where $f(X,Y)=X^2-dY^2$ can have only finitely many integer solutions due to the factorization $f(X,Y)=(X-mY)(X+mY)$. In the case of $d=p^{2s}$, consider equation (1) given here as $x(x+1)=dy(y+1)$. If one assumes to the contrary that a positive integer solution (x, y) exists, then $p^{2s} \mid x(x+1)$. However, this can only be true if either $p^{2s} \mid x$ or $p^{2s} \mid (x+1)$ as $(x, x+1)=1$. Suppose $x = mp^{2s}$ for some fixed $m \in N\setminus\{0\}$, then y must be a root of the quadratic $0=y^2+y-(m^2p^{2s}+m)$. However, as the discriminant of this equation satisfies the inequality

$$(2p^sm)^2 < 4p^{2s}m^2+4m+1<(2p^sm+1)^2$$

and so cannot be a square, one deduces that $y \notin N$. A similar contradiction follows if $x+1 = mp^{2s}$, as the discriminant of the resulting quadratic satisfies the inequality

$$(2p^sm-1)^2 < 4p^{2s}m^2-4m+1<(2p^sm)^2. \square$$

Remark 2.2: One can use the above argument to compute an infinite subset of solutions to the Diophantine equation $x(x+m)=dy(y+m)$ for nonsquare d via (5). All that is required is the determination of the element $\tau_\Delta = u+v\rho_\Delta$, which will result upon finding the unit $\varepsilon \in \mathcal{O}_{\Delta,+}^{\times}$. This can be achieved by applying the following method taken from [2]. Consider the quadratic form $f_{4d}(x,y)=x^2-dy^2$, which has a nonsquare discriminant $4d>0$. If y is the smallest positive integer such that one of the dy^2+1 or dy^2-1 is a square and x is the positive integer root, then $\varepsilon = x+y\sqrt{d}$.

When determining the full solution set one will, of course, have to find all the distinct orbits that comprise $\mathcal{S}$. This can be achieved because a finite list containing the generators of each such orbit can be constructed using the following result (see [2]).

Proposition 2.1: Let $f(x,y)$ be an integral form with discriminant Δ and suppose $m \neq 0 \in Z$. If $\tau = \tau_\Delta$ is the smallest unit in $\mathcal{O}_{\Delta,1}^{\times}$ that is greater than unity, then:

(i) Every orbit of integral solutions of $f(x,y)=m$ contains a solution $(x,y)\in Z^2$ such that $0 \le y \le U$, where $U = |am\tau/\Delta|^{1/2}(1-1/\tau)$ if $am>0$ and $U=|am\tau/\Delta|^{1/2}(1+1/\tau)$ if $am<0$.

(ii) Two distinct solutions $(x_1,y_1)\neq(x_2,y_2)\in Z^2$ of the equation $f(x,y)=m$ such that $0\le y_i \le U$ belong to the same orbit if and only if $y_1=y_2=0$ or $y_1=y_2=U$.

Since every orbit of solutions to $f(x,y)=m$ contains an element in the finite set

$$\mathcal{S}' = \{(x,y):\ 0\le y\le U\},$$

any $(x, y) \in \mathcal{S}'$ can be listed and sorted into orbits using Proposition 2.1. The set $\mathcal{S}'$ which contains the representatives of the orbits can be viewed as a finite list from which the solutions in $\mathcal{S}$ can be generated from the group action. In the case of (6), it is of interest to estimate the maximum number of distinct orbits needed to describe $\mathcal{S}$ as $d \to \infty$. Using the following result, which is taken from [4], we can obtain an asymptotic bound for the maximum number of possible orbits for the Diophantine equation in (6).

Lemma 2.1: If $\tau_\Delta = u + \rho_\Delta v$ is the smallest unit in $\mathcal{O}_{\Delta,1}^\times$ that is greater than unity with $\Delta = 4d$, then $u \sim \sqrt{d}e^{\sqrt{d}+O(1)}$ as $d \to \infty$ through nonsquare values.

Theorem 2.2: For a given $m \in Z \setminus \{0\}$, the maximum number of distinct orbits for the equation $x(x+m) = dy(y+m)$ is given by $[M(d)]$, where

$$M(d) \sim \frac{|k|d^{1/4}}{\sqrt{2}}e^{\frac{\sqrt{d}}{2}+O(1)}$$

as $d \to \infty$ through nonsquare values where $k = m/2$ for m even and $k = m$ for k odd.

Proof: We first note that no proper orbit can be generated from the solution $(0, 0)$. Thus, from Proposition 2.1, the maximum number of distinct orbits is equal to the total number of positive integers less than or equal to U, that is, $[U]$. Now when $m = 2s$ and d is nonsquare, the solutions of the Diophantine equation in (2) arise directly from the orbits of $Z^2 - dW^2 = s^2(1-d)$ via a translation of these orbits by subtraction of the ordered pair (s, s). Consequently, we have by Lemma 2.1 that

$$\begin{aligned} M(d) &= \left|\frac{s^2(1-d)}{4d}\right|^{1/2}\left|\tau_\Delta\left(1+\frac{1}{t_\Delta}\right)^2\right|^{1/2} = \frac{|s|}{2}\left|\frac{1-d}{d}\right|^{1/2}|\tau_\Delta + 2 + \sigma(\tau_\Delta)|^{1/2} \\ &= \frac{|s|}{\sqrt{2}}\left|\frac{1-d}{d}\right|^{1/2}|u+1|^{1/2} \sim \frac{|s|}{\sqrt{2}}(\sqrt{d}e^{\sqrt{d}+O(1)})^{1/2} \end{aligned}$$

as $d \to \infty$. When m is odd, the solutions are derived from the odd solutions in the orbits of $X^2 - dY^2 = m^2(1-d)$. Thus, if in each of these orbits there exists an infinite subset of odd solutions, then the maximum number of distinct orbits is again $[U]$ and the asymptotic bound will result as in the above by replacing s by m. □

3. AN ELEMENTARY APPROACH

In contrast to the algebraic methods used previously, we present in this section an alternate technique for demonstrating the solvability of (1) for the cases $d = 2, 3$. Although of interest on its own, the elementary approach employed here has the advantage of allowing one to deduce a characterization for the solutions of the negative Pell equation in terms of square triangular numbers. We first observe that, if $0 < x \le y$, then $x(x+1) < dy(y+1)$, while, if $x \ge dy > 0$, then $x(x+1) > dy(y+1)$. Consequently, for an arbitrary $y \in N \setminus \{0\}$, the only integer values which x may possibly assume in order that $x(x+1) = dy(y+1)$ are those in which $y < x < dy$. So, if (x, y) is a solution, then there must exist a fixed $t \in N \setminus \{0\}$ such that $x = y + t$ and $(y+t)(y+t+1) = dy(y+1)$. With the introduction of the parameter t, one can then solve for y in terms of t and so the question of solvability is necessarily reduced to knowing whether the discriminant of the associated quadratic is a square for infinitely many t. We now apply this method of the case $d = 3$. The following technical lemma will be required to establish the necessary condition for the existence of integer solutions.

Lemma 3.1: Suppose (v, u) is a positive integer solution of the Pell equation $v^2 - 3u^2 = 1$. Then $(v, u) = (2, 1)^n$, for some $n \geq 1$, where the product of solutions is taken in the sense of (4).

Proof: Applying the method in Remark 2.2, one deduces for $\Delta = 12$ that the element $\varepsilon \in \mathcal{O}_{\Delta,+}^{\times}$ is given by $\varepsilon = 2+\sqrt{3}$. As $N(\varepsilon) = 1$, we have $\tau_{12} = \varepsilon$ and so $Pell(\Delta) = \{\pm(x_n, y_n):$ $n \in Z\}$, where $(x_n + y_n\sqrt{3}) = (2+\sqrt{3})^n$. Thus, the positive solutions are given by (x_n, y_n), where $n \in N \setminus \{0\}$. Now, since $(x_{n+1}, y_{n+1}\sqrt{3}) = (x_n + y_n\sqrt{3})(2+\sqrt{3})$, one sees that $x_{n+1} = 2x_n + 3y_n$ and $y_{n+1} = x_n + 2y_n$. Consequently, via the product formula for solutions in (4) we have $(x_{n+1}, y_{n+1}) =$ $(x_n, y_n)\cdot(2, 1)$, from which it is deduced that $(x_n, y_n) = (2, 1)^n$ as $(x_1, y_1) = (2, 1)$. □

Theorem 3.1: There are infinitely many positive integer solutions to (1) in the case $d = 3$. Moreover, all such solutions (x, y) are given by

$$\left(\frac{3u_n + v_n - 1}{2}, \frac{u_n + v_n - 1}{2}\right),$$

where the ordered pair (v_n, u_n) is generated recursively using

$$\begin{pmatrix} v_{i+1} \\ u_{i+1} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_i \\ u_i \end{pmatrix}, \tag{10}$$

with $(v_1, u_1) = (2, 1)$.

Proof: We first prove existence. Suppose (x, y) satisfies the Diophantine equation, then by the above method there must exist a fixed $t > 0$ such that $x = y + t$. Substituting this expression into the Diophantine equation and simplifying yields the quadratic $0 = 2y^2 + 2(1-t)y - (t^2 + t)$. Remembering that y is assumed positive, one finds upon solving this equation that

$$y = \frac{t - 1 + \sqrt{3t^2 + 1}}{2}. \tag{11}$$

However, from Lemma 3.1, there are infinitely many $s, t \in N$ such that $3t^2 + 1 = s^2$; moreover, by a simple parity argument, the numerator in (11) can be shown to be an even integer for all such t. Consequently, there are infinitely many integers (x, y) that satisfy $s(s+1) = 3y(y+1)$, all of which may be determined via (11). It is now a simple task to construct the accompanying algorithm. Set $t = v_n$ and $s = u_n$ as in Lemma 3.1, then, clearly, from (11) we have

$$y = \frac{v_n - 1 + \sqrt{u_n^2}}{2} = \frac{v_n + u_n - 1}{2}, \quad x = v_n + y = \frac{3v_n + u_n - 1}{2}.$$

Finally, as $(v_{n+1}, u_{n+1}) = (2, 1)\cdot(2, 1)^n = (2, 1)\cdot(v_n, u_n)$, one deduces from the product formula in (4) the recurrence relation of (10). □

For larger values of d, the above method cannot be applied due to the increased difficulty in verifying the existence of infinitely many t for which the discriminant of $(y+t)(y+t+1) =$ $dy(y+1)$ is a square. To conclude, we examine an application of our elementary method to uncover a curious connection between the solutions of the Diophantine equation $X^2 - 2Y^2 = -1$ and the sequence of square triangular numbers. Following the above analysis it is easily seen, in the case $d = 2$, that for (x, y) to be a positive integer solution of $x(x+1) = 2y(y+1)$ there must exist a $t > 0$ such that $x = y + t$ with

$$y = \frac{2t - 1 + \sqrt{8t^2 + 1}}{2}.$$

Since y is an integer, $8t^2 + 1$ must be an odd perfect square. Consequently, we require $8t^2 + 1 = (2m+1)^2$ for some $m \in N \setminus \{0\}$, so that t and m satisfy $2t^2 = m(m+1)$. Thus, by denoting T_n as the square root of the n^{th} square triangular number, of which there are infinitely many, one deduces for some $n \in N \setminus \{0\}$ that

$$x = \frac{4T_n - 1 + \sqrt{8T_n^2 + 1}}{2}, \quad y = \frac{2T_n - 1 + \sqrt{8T_n^2 + 1}}{2}. \tag{12}$$

Using these relations, one can deduce the following characterization.

Theorem 3.2: All positive integer solutions (X, Y) of the negative Pell equation $X^2 - 2Y^2 = -1$ are of the form

$$(4T_n + \sqrt{8T_n^2 + 1},\ 2T_n + \sqrt{8T_n^2 + 1}),$$

where T_n denotes the positive square root of the n^{th} square triangular number.

Proof: Recall that the negative Pell equation $X^2 - 2Y^2 = -1$, where $X = 2x + 1$, $Y = 2y + 1$, can be derived by completing the square on $x(x+1) = 2y(y+1)$. The result will follow from (12) if one can show that all the positive solutions (X, Y) consist only of odd integers. To establish this, we first observe from Remark 2.2 that, for $\Delta = 8$, the element $\varepsilon \in \mathbb{O}_{\Delta,+}^{\times}$ is given by $\varepsilon = 1 + \sqrt{2}$. Therefore, $Pell^{\pm}(8) = \{\pm(x_n, y_n) : n \in Z\}$, where $x_n + y_n\sqrt{2} = (1+\sqrt{2})^n$. However, since $N(1+\sqrt{2}) = -1$, the positive solutions of the negative Pell equation must be given by $(X_n, Y_n) = (x_{2n+1}, y_{2n+1})$, where $n \in N$. Moreover, since $x_{2n+3} + y_{2n+3}\sqrt{2} = (1+\sqrt{2})^2(x_{2n+1} + y_{2n+1}\sqrt{2})$, one can see that the solutions (X_n, Y_n) satisfy the recurrence relation

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, \text{ with } (X_0, Y_0) = (1, 1).$$

The desired conclusion follows now by a simple inductive argument. □

REFERENCES

1. E. Becker, R. Robson, G. Schrange. "Variationen über ein diophantisches Thema." *El. Math.* **50** (1995):45-64.
2. D. E. Flath. *Introduction to Number Theory*. New York: Wiley & Sons, 1988.
3. M. A. Nyblom. "A Note on the Set of Almost Isosceles Right-Angled Triangles." *The Fibonacci Quarterly* **36.4** (1998):319-22.
4. A. Peretti. "The Diophantine Equation $t^2 - Dy^2 = 1$." *Bull. Number Theory* **13** (1989):61-68.

AMS Classification Numbers: 11D09, 11E10

❖❖❖