APPROXIMATION OF ∞-GENERALIZED FIBONACCI SEQUENCES AND THEIR ASYMPTOTIC BINET FORMULA

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1. INTRODUCTION

The notion of an ∞-generalized Fibonacci sequence has been introduced and studied in [8], [9], and [11]. In fact, such a notion goes back to Euler. In his book [4], he discusses Bernoulli's method of using linear recurrences to approximate roots of (mainly polynomial) equations. At the very end, in Article 355 [4, p. 301], there is a brief example of the use of an ∞-generalized Fibonacci sequence for the approximation of a root of a power series equation.*

The class of sequences defined by linear recurrences of infinite order is an extension of the class of ordinary *r-generalized Fibonacci sequences* (r-GFS, for short) with r finite defined by linear recurrences of r^{th} order (for example, see [1], [2], [3], [6], [7], [10], etc.). More precisely, let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be two sequences of real or complex numbers, where $a_j \neq 0$ for some j. The former is called the *coefficient sequence* and the latter the *initial sequence*. The associated *∞-generalized Fibonacci sequence* (∞-GFS, for short) $\{V_n\}_{n\in\mathbb{Z}}$ is defined as follows:

$$V_n = \alpha_n \quad (n \leq 0), \tag{1.1}$$

$$V_n = \sum_{j=0}^{\infty} a_j V_{n-j-1} \quad (n \geq 1). \tag{1.2}$$

As is easily observed, the general terms V_n may not necessarily exist. In [8], a sufficient condition for the existence of the general terms has been given.

In this paper, we first give a necessary and sufficient condition for the existence of the general terms V_n $(n \geq 1)$ of an ∞-GFS (see Section 2). We will see that the condition in [8] satisfies our condition, but not vice versa. We then consider a process of approximating a given ∞-GFS by a sequence of r-GFS's, where $r < \infty$ varies (see Section 3). As is well known, there is a Binet-type

* The authors would like to thank the referee for kindly pointing out Euler's work.

formula for the general terms of an r-GFS (for example, see Theorem 1 in [3]). In Section 4, we use such a formula together with the approximation result in Section 3 to obtain an asymptotic Binet formula for an ∞-GFS. In Section 5, we study the asymptotic behavior of ∞-GFS's using the results in the previous sections. In Section 6, we concentrate on the case in which $a_j \geq 0$ and obtain some sharp results about the asymptotic behavior of ∞-GFS's. Finally, in Section 7, we give an explicit example of our main theorem of Section 6.

2. EXISTENCE OF GENERAL TERMS

Let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be as in Section 1 and $\{V_n\}_{n\in \mathbf{Z}}$ be the associated ∞-GFS defined by (1.1) and (1.2). Equation (1.2) can be rewritten as follows:

$$V_n = \sum_{j=0}^{n-2} a_j V_{n-j-1} + \sum_{j=n-1}^{\infty} a_j V_{n-j-1} = \sum_{j=0}^{n-2} a_j V_{n-j-1} + \sum_{j=0}^{\infty} a_{j+n-1}\alpha_{-j}. \tag{2.1}$$

Then it is easy to see that we have the following necessary and sufficient condition for the existence of the general terms V_n $(n \geq 1)$.

Proposition 2.1: The general term V_n exists for all $n \geq 1$ if and only if the following condition (C_∞) is satisfied.

(C_∞): The series $\sum_{j=0}^{\infty} a_{j+n-1}\alpha_{-j}$ converges for all $n \geq 1$.

Condition (C_∞) is trivially satisfied in the case of an r-GFS with r finite, since $a_j = 0$ for all $j \geq r$.

Remark 2.2: As particular cases of Proposition 2.1, we can easily prove the following.

- ***(a)*** If the series $\sum_{j=0}^{\infty} \alpha_{-j}$ converges absolutely and the sequence $\{a_j\}_{j\geq 0}$ is bounded, then V_n exists for all $n \geq 1$.
- ***(b)*** If the series $\sum_{j=0}^{\infty} a_j$ converges absolutely and the sequence $\{\alpha_{-j}\}_{j\geq 0}$ is bounded, then V_n exists for all $n \geq 1$.

For another existence result, see Lemma 6.6. Compare Remark 2.2 with Section 2.1 in [11].

Now let us compare our condition (C_∞) with the sufficient condition considered in [8] for the existence of the general terms V_n $(n \geq 1)$. Let $h(z)$ be the power series defined by $h(z) = \sum_{j=0}^{\infty} a_j z^j$. The conditions considered in [8] are the following.

$(C1)$: The radius of convergence R of the power series $h(z)$ is positive.

$(C2)$: There exist $C > 0$ and $T > 0$ with $0 < T < R$ satisfying $|\alpha_{-j}| \leq CT^j$ for all $j \geq 0$.

It was established in [8] that, if conditions $(C1)$ and $(C2)$ are satisfied, then the general term V_n of the associated ∞-GFS exists for all $n \geq 1$.

It is easy to see that, if conditions $(C1)$ and $(C2)$ are satisfied, then (C_∞) is also satisfied. On the other hand, the examples $a_j = (j+1)^{-3}$, $\alpha_{-j} = j$, and $a_j = (j+1)^{-1}$, $\alpha_{-j} = (-1)^j$ both satisfy condition $(C1)$, but not $(C2)$, while (C_∞) is satisfied in both cases. Therefore, condition (C_∞) is strictly weaker than $(C1)$ and $(C2)$.

3. APPROXIMATION BY r-GFS's WITH r FINITE

Let $\{a_j\}_{j\ge 0}$ and $\{\alpha_{-j}\}_{j\ge 0}$ be sequences of complex numbers as before. For each $r\ge 1$, let $\{V_n^{(r)}\}_{n\ge -r+1}$ be the r-GFS defined as follows:

$$V_n^{(r)}=\alpha_n \quad (n=-r+1,\ -r+2,\dots,0), \tag{3.1}$$

$$V_n^{(r)}=\sum_{j=0}^{r-1}a_jV_{n-j-1}^{(r)} \quad (n\ge 1). \tag{3.2}$$

Note that here we allow the case where $a_{r-1}=0$, while $a_{r-1}\neq 0$ is assumed in [3].

In this section, we prove the following approximation theorem.

Theorem 3.1: The general term V_n exists for all $n\ge 1$ if and only if the sequence $\{V_n^{(r)}\}_{r\ge 1}$ converges for all $n\ge 1$. Furthermore, in this case, for all $n\ge 1$, we have

$$V_n=\lim_{r\to\infty}V_n^{(r)}. \tag{3.3}$$

Proof: We prove, by induction on k, that the terms $V_1,\dots,V_k$ exist if and only if, for all n with $1\le n\le k$, the sequence $\{V_n^{(r)}\}_{r\ge 1}$ converges and (3.3) holds. When $k=1$, we have

$$V_1=\sum_{j=0}^{\infty}a_j\alpha_{-j} \quad\text{and}\quad V_1^{(r)}=\sum_{j=0}^{r-1}a_j\alpha_{-j}$$

for all $r\ge 1$. Thus, V_1 exists if and only if the sequence $\{V_1^{(r)}\}_{r\ge 1}$ converges. Furthermore, in this case, we have $V_1=\lim_{r\to\infty}V_1^{(r)}$.

Now suppose $k\ge 2$ and that the induction hypothesis holds for $k-1$. For $r\ge k$, we have

$$V_k=\sum_{j=0}^{k-2}a_jV_{k-j-1}+\sum_{j=k-1}^{\infty}a_j\alpha_{k-j-1}$$

and

$$V_k^{(r)}=\sum_{j=0}^{k-2}a_jV_{k-j-1}^{(r)}+\sum_{j=k-1}^{r-1}a_j\alpha_{k-j-1}. \tag{3.4}$$

Then, by our induction hypothesis, we see that the sequence $\{V_n^{(r)}\}_{r\ge 1}$ converges for all n with $1\le n\le k$ if and only if the terms $V_1,\dots,V_k$ exist. Furthermore, in this case, using our induction hypothesis, we see that (3.3) holds for $n=k$ by sending $r\to\infty$ in (3.4). □

4. ASYMPTOTIC BINET FORMULA

Let $\{a_j\}_{j\ge 0}$ and $\{\alpha_{-j}\}_{j\ge 0}$ be sequences of complex numbers. For each $r\ge 1$, consider the polynomial $Q_r(z)$ defined by

$$Q_r(z)=1-\sum_{j=0}^{r-1}a_jz^{j+1}. \tag{4.1}$$

Note that the characteristic polynomial $P_r(z)$ of the r-GFS $\{V_n^{(r)}\}_{n\ge -r+1}$ defined by (3.1) and (3.2) is given by

$$P_r(z)=z^rQ_r(z^{-1}), \tag{4.2}$$

which is a polynomial of degree r. Let $\lambda_1^{(r)}, \ldots, \lambda_{u(r)}^{(r)}$ be the complex roots of $P_r(z)$, whose respective multiplicities are $m_1^{(r)}, \ldots, m_{u(r)}^{(r)}$. Note that $m_1^{(r)} + \cdots + m_{u(r)}^{(r)} = r$. The classical Binet-type formula for the r-GFS $\{V_n^{(r)}\}_{n \geq -r+1}$ is given by the following:

$$V_n^{(r)} = \sum_{k=1}^{u(r)} \sum_{j=0}^{m_r^{(k)}-1} \beta_{k,j}^{(r)} n^j (\lambda_k^{(r)})^n, \tag{4.3}$$

where the complex numbers $\beta_{k,j}^{(r)}$ are determined by the initial sequence $\{\alpha_{-j}\}_{0 \leq j \leq r-1}$ (e.g., see [5, Theorem 3.7]; [3, Theorem 1]).

Remark 4.1: In [5] and [3] it is assumed that $a_{r-1} \neq 0$. When this condition is not satisfied, the polynomial $Q_r(z)$ may not necessarily be of degree r. On the other hand, the characteristic polynomial $P_r(z)$ is always of degree r, which may have zero as a root of some multiplicity. Hence, the above Binet-type formula (4.3) holds even if $a_{r-1} = 0$.

By Proposition 2.1, Theorem 3.1, and (4.3), we have the following asymptotic Binet formula.

Theorem 4.2: If condition (C_∞) is satisfied, then we have, for all $n \geq 1$,

$$V_n = \lim_{r \to \infty} \sum_{k=1}^{u(r)} \sum_{j=0}^{m_k^{(r)}-1} \beta_{k,j}^{(r)} n^j (\lambda_k^{(r)})^n. \tag{4.4}$$

Compare the above results with Problem 4.5 in [8].

Example 4.3: Consider the ∞-GFS $\{V_n\}_{n \in \mathbb{Z}}$ associated with the coefficient sequence $a_j = -\gamma^{j+1}$ and the initial sequence $\alpha_{-j} = \delta_{0j}$ $(j \geq 0)$, where γ is a nonzero complex number, $\delta_{0j} = 0$ if $j \neq 0$, and $\delta_{00} = 1$. Note that condition (C_∞) is trivially satisfied. By a straightforward calculation, we see that

$$V_n = \begin{cases} 0 & (n \neq 0, 1), \\ 1 & (n = 0), \\ -\gamma & (n = 1). \end{cases} \tag{4.5}$$

On the other hand, we have $P_r(z) = z^r + \gamma z^{r-1} + \cdots + \gamma^{r-1} z + \gamma^r$. Thus, all the roots are simple and they are of the form $\lambda_k^{(r)} = \gamma \xi_{r+1}^k$ $(k = 1, 2, \ldots, r)$ for a primitive $(r+1)^{\text{st}}$ root ξ_{r+1} of unity. Then we have*

$$\sum_{k=1}^{r} \beta_{k,0}^{(r)} (\lambda_k^{(r)})^n = \delta_{0n} \quad (-r+1 \leq n \leq 0). \tag{4.6}$$

We multiply each of the equations of (4.6) by γ^{-n} and sum them up for $n = -r+1, \ldots, 0$. Then we obtain

$$\sum_{k=1}^{r} \beta_{k,0}^{(r)} (\lambda_k^{(r)})^{-r} = -\gamma^{-r}, \tag{4.7}$$

since

$$\sum_{n=-r+1}^{0} (\lambda_k^{(r)})^n \gamma^{-n} = -(\lambda_k^{(r)})^{-r} \gamma^r.$$

* Using (4.6), we can obtain explicit values of $\beta_{k,0}^{(r)}$, although we do not need them here.

By successively multiplying (4.6) and (4.7) by $\gamma^{r+1} = (\lambda_k^{(r)})^{r+1}$, we see that

$$V_n^{(r)} = \begin{cases} 0, & n \not\equiv 0,1 \pmod{r+1}, \\ \gamma^n, & n \equiv 0 \pmod{r+1}, \\ -\gamma^n, & n \equiv 1 \pmod{r+1}, \end{cases} \tag{4.8}$$

by (4.3). Hence, we have $\lim_{r\to\infty} V_n^{(r)} = V_n$ in view of (4.5).

5. ASYMPTOTIC BEHAVIOR OF ∞-GFS's

Let $\{a_j\}_{j\geq 0}$ and $\{\alpha_{-j}\}_{j\geq 0}$ be sequences of complex numbers. For each $r \geq 1$, consider the characteristic polynomial $P_r(z)$ of the r-GFS $\{V_n^{(r)}\}_{n\geq -r+1}$ as in (4.2). Let $r_0 \geq 1$ be an integer such that $a_{r_0-1} \neq 0$ and let us assume that, for each $r \geq r_0$, there exists a nonzero *dominant root* q_r of $P_r(z)$ with *dominant multiplicity* 1 (for these terminologies, refer to Section 3 in [3]). In [3], it has been shown that $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ exists and its explicit value has been obtained in terms of q_r together with the coefficient and the initial sequences.

Let us assume that the sequence $\{q_r\}_{r\geq r_0}$ converges to a nonzero complex number q. If one looks at Theorem 4.2, then it might seem easy to obtain a convergence result for the sequence $\{V_n/q^n\}_{n\geq 1}$. However, since equation (4.4) is given by the limit for $r\to\infty$, we have to be careful with the relationship between the convergence with respect to r and that with respect to n. For this reason, we need the following definition.

Definition 5.1: Let $\{x_n^{(r)}\}_{n\geq n_0, r\geq r_0}$ be a doubly-indexed sequence of real or complex numbers. We say that the sequences $\{x_n^{(r)}\}_{n\geq n_0}$ are *uniformly convergent* for $r \geq r_0$ if there exists a sequence $\{L_r\}_{r\geq r_0}$ of real or complex numbers such that, for every $\varepsilon > 0$, there exists an $N \geq n_0$ satisfying $|x_n^{(r)} - L_r| < \varepsilon$ for all $n \geq N$ and all $r \geq r_0$. It is easy to see that in this case, if the sequence $\{x_n^{(r)}\}_{r\geq r_0}$ converges to x_n for each $n \geq n_0$, and if $L = \lim_{r\to\infty} L_r$ exists, then $\lim_{n\to\infty} x_n$ exists and is equal to L.

Then, combining the results of [3], Theorem 3.1 of the present paper, and the above definition, we obtain the following (for an explicit example, see Section 7).

Theorem 5.2: Suppose that

(a) $P_r(z)$ has a nonzero dominant root q_r of dominant multiplicity 1 for each $r \geq r_0$,
(b) $q = \lim_{r\to\infty} q_r$ exists and is nonzero,
(c) the general term V_n exists for all $n \geq 1$,
(d) the sequences $\{x_n^{(r)}\}_{n\geq 0} = \{V_n^{(r)}/q_r^n\}_{n\geq 0}$ are uniformly convergent for $r \geq r_0$ with $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$, and
(e) $L = \lim_{r\to\infty} L_r$ exists.

Then the limit $\lim_{n\to\infty} V_n/q^n$ exists and is equal to L.

Proof: By Theorem 3.1 and our assumptions, we have $V_n/q^n = \lim_{r\to\infty} V_n^{(r)}/q_r^n$ for each $n \geq 1$. Then, by the observation given in Definition 5.1 together with our assumptions, we have $\lim_{n\to\infty} V_n/q^n = L$. □

Remark 5.3: As in the above theorem, let us assume (a)-(c) and, instead of (d) and (e), let us assume that $L=\lim_{n,r\to\infty} x_n^{(r)}$ exists, where we write $\lim_{n,r\to\infty} x_n^{(r)}=L$ if, for every $\varepsilon>0$, there exists an $N\geq r_0$ such that $|x_n^{(r)}-L|<\varepsilon$ for all $n,r\geq N$. Then we have

$$L=\lim_{n\to\infty}\frac{V_n}{q^n}=\lim_{r\to\infty}L_r. \tag{5.1}$$

The following lemma is easy to prove.

Lemma 5.4: Let $\{y_n^{(r)}\}_{n\geq n_0, r\geq r_0}$ be a doubly-indexed sequence of real or complex numbers such that, for every $n\geq n_0$, $\lim_{r\to\infty} y_n^{(r)}=\gamma_n$ exists and $\lim_{n\to\infty}\gamma_n=\gamma$ exists. Then, for every $n\geq n_0$, there exists an $r(n)\geq r_0$ such that $r(n)<r(n+1)$ for all $n\geq n_0$ and that the sequence $\{y_n^{(r(n))}\}_{n\geq n_0}$ converges to γ.

Let us assume conditions (a)-(c) of Theorem 5.2 and, for $n\geq 1$ and $r\geq r_0$, set $y_n^{(r)}=V_n/q^n-V_n^{(r)}/q_r^n$. Then, for every $n\geq 1$, we have $\lim_{r\to\infty} y_n^{(r)}=\gamma_n=0$. Then $\lim_{n\to\infty}\gamma_n=0$ trivially exists. Thus, Lemma 5.4 implies that, for every $n\geq 1$, there exists an $r(n)\geq r_0$ such that $r(1)<r(2)<r(3)<\cdots$ and $\lim_{n\to\infty} y_n^{(r(n))}=0$. Therefore, we have the following theorem.

Theorem 5.5: Suppose that

- ***(a)*** $P_r(z)$ has a nonzero dominant root q_r of dominant multiplicity 1 for each $r\geq r_0$,
- ***(b)*** $q=\lim_{r\to\infty} q_r$ exists and is nonzero, and
- ***(c)*** the general term V_n exists for all $n\geq 1$.

Then $L=\lim_{n\to\infty} V_n/q^n$ exists if and only if $\lim_{n\to\infty} V_n^{(r(n))}/q_{r(n)}^n$ exists. Furthermore, in this case, we have

$$L=\lim_{n\to\infty}\frac{V_n}{q^n}=\lim_{n\to\infty}\frac{V_n^{(r(n))}}{q_{r(n)}^n}. \tag{5.2}$$

In (5.1) and (5.2), we did not give the limiting value L explicitly. In the following section, we determine the explicit value in the case where a_j are nonnegative real numbers.

6. THE CASE OF NONNEGATIVE COEFFICIENTS

In this section, we assume that all the coefficients a_j are nonnegative real numbers and consider the same problem as in the previous section. We use the same notations.

It is not difficult to see that, for each $r\geq r_0$, there always exists a unique real number $q_r>0$ such that $P_r(q_r)=Q_r(q_r^{-1})=0$ (for example, see Lemma 2 in [2], Lemma 8 in [3], and Section 12 in [12]), where Q_r is the polynomial defined by (4.1). Set $p_r=q_r^{-1}$. Define the power series $Q(z)$ by $Q(z)=1-zh(z)=1-\sum_{j=0}^{\infty} a_j z^{j+1}$ and let R be the radius of convergence of $Q(z)$, which coincides with that of $h(z)$. The following will be proved later in this section.

Theorem 6.1: The sequence $\{q_r^{-1}\}_{r\geq r_0}=\{p_r\}_{r\geq r_0}$ always converges and the following conditions are equivalent:

- ***(a)*** Condition $(C1)$ is satisfied (i.e., $R>0$) and $\lim_{x\to R-0} Q(x)\leq 0$.
- ***(b)*** The limiting value $l=\lim_{r\to\infty} p_r>0$ and $Q(l)=0$.
- ***(c)*** There exists a unique positive real number p such that $Q(p)=0$.

Furthermore, if (c) is satisfied, then we have $p=\lim_{r\to\infty} p_r$.

The main result of this section is the following theorem.

Theorem 6.2: Assume that one of the three conditions of Theorem 6.1 is satisfied. Suppose that $d_{r_1}=1$ for some $r_1 \geq r_0$, $0<p<R$, and

$$q^j|\alpha_{-j}|<K \quad (j \geq 0) \tag{6.1}$$

for some constant $K>0$, where $d_{r_1}=\gcd\{j+1: a_j>0, 0 \leq j \leq r_1-1\}$ and $q=p^{-1}$. If the sequences $\{V_n^{(r)}/q_r^n\}_{n\geq 1}$ are uniformly convergent for $r \geq r_1$, then V_n exists for all n and we have

$$\lim_{n\to\infty}\frac{V_n}{q^n}=\frac{\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}a_kq^{j-k-1}\right)\alpha_{-j}}{\sum_{j=0}^{\infty}(j+1)a_jq^{-(j+1)}}. \tag{6.2}$$

Let us begin by proving Theorem 6.1.

Proof of Theorem 6.1: Suppose that $r_0 \leq r<r'$. Then we have $Q_{r'}(p_r)=-a_rp_r^{r+1}-\cdots-a_{r'-1}p_r^{r'} \leq 0$. Furthermore, we have $Q_{r'}(p_{r'})=0$. Since $Q_{r'}(x)$ is a decreasing function on $(0,\infty)$, we have $p_r \geq p_{r'}$; i.e., the sequence $\{p_r\}_{r\geq r_0}$ of positive real numbers is nonincreasing. Hence, it is convergent. In the following, we set $l=\lim_{r\to\infty}p_r \geq 0$.

For every $r \geq r_0$, we have $0 \leq l \leq p_r$. Since $Q_r(x)$ is a decreasing function on $(0,\infty)$, we have $0 \leq Q_r(l) \leq 1$. On the other hand, since $Q_{r'}(l)-Q_r(l)=-a_rl^{r+1}-\cdots-a_{r'-1}l^{r'} \leq 0$ for r, $r' \geq r_0$ with $r<r'$, we see that the sequence $\{Q_r(l)\}_{r\geq r_0}$ is nonincreasing. Thus, $\lim_{r\to\infty}Q_r(l)$ exists and is equal to $Q(l)$. Furthermore, we have

$$0 \leq Q(l) \leq 1. \tag{6.3}$$

(a) $\Rightarrow$ (b): First, note that since $Q(l)$ exists we have $0 \leq l \leq R$.

Suppose $0 \leq l<R$ and $Q(l)>0$. Since $Q(x)$ is a continuous function on the interval $(-R,R)$, there exists a sufficiently small positive real number η such that $Q(x)>0$ for all $x \in (l-\eta, l+\eta) \subset (-R,R)$. Since $l=\lim_{r\to\infty}p_r$, there exists an $r' \geq r_0$ such that $p_r \in [l, l+\eta)$ for all $r \geq r'$. Thus, $Q(p_r)>0$ for all $r \geq r'$. However, since $Q(p_r)=-\sum_{j=r}^{\infty}a_jp_r^{j+1} \leq 0$, this is a contradiction. Therefore, we have $Q(l)=0$.

If $l=R$, then we have $0 \leq Q(R) \leq 1$ by (6.3). Thus, we have $Q(R)=Q(l)=0$, since $Q(R)=\lim_{x\to R-0}Q(x) \leq 0$ by our assumption.

Therefore, we have $Q(l)=0$, and this implies that $l>0$, since, if $l=0$, we would have $Q(l)=1>0$.

(b) $\Rightarrow$ (c): Setting $p=l$, we have $Q(p)=0$. The uniqueness follows from the fact that $Q(x)$ is a strictly decreasing function.

(c) $\Rightarrow$ (a): Since $p>0$ and $Q(p)=0$, we see that $0<p \leq R$, which implies condition $(C1)$. Furthermore, since $Q(x)$ is a decreasing function on $(0,R)$, we have $\lim_{x\to R-0}Q(x) \leq Q(p)=0$. This completes the proof. □

Remark 6.3: When some a_j is not a nonnegative real number, there does not always exist a root p of $Q(z)$. For instance, in Example 4.3 of Section 4, we have $Q(z)=1/(1-\gamma z)$, which never takes the value zero inside the convergence range. Compare this observation with Problem 4.5 in [8].

Since q_r is a root of the characteristic polynomial P_r, we have

$$\frac{a_0}{q_r}+\frac{a_1}{q_r^2}+\cdots+\frac{a_{r-1}}{q_r^r}=1. \tag{6.4}$$

Combining this with Theorems 3, 5, and 9 of [3], we have the following lemma.

Lemma 6.4: For each $r \ge r_0$, we have:

(a) $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ exists for any initial values $\{\alpha_{-j}\}_{0\le j\le r-1}$ and is nonzero for some initial values if and only if $d_r = 1$.

(b) If there exists an $r_1 \ge r_0$ such that $d_{r_1}=1$, then $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ exists for all $r \ge r_1$. Furthermore, this limit is given by

$$L_r = \frac{\sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1} a_k q_r^{j-k-1}\right)\alpha_{-j}}{\sum_{j=0}^{r-1}(j+1)a_j q_r^{-(j+1)}}. \tag{6.5}$$

Lemma 6.5: Assume that one of the three conditions of Theorem 6.1 is satisfied. Suppose that $d_{r_1}=1$ for some $r_1 \ge r_0$, $0<p<R$, and (6.1) holds for some constant $K>0$. Then, for $L_r = \lim_{n\to\infty} V_n^{(r)}/q_r^n$ $(r \ge r_1)$, we have

$$\lim_{r\to\infty} L_r = \frac{\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k q^{j-k-1}\right)\alpha_{-j}}{\sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)}} < +\infty. \tag{6.6}$$

Proof: Set $S_r(x) = \sum_{j=0}^{r-1}(j+1)a_j x^{j+1}$. Since $0<p=q^{-1}\le p_r = q_r^{-1}$ for all $r \ge r_0$, we have

$$S_r(q^{-1}) = \sum_{j=0}^{r-1}(j+1)a_j q^{-(j+1)} \le \sum_{j=0}^{r-1}(j+1)a_j q_r^{-(j+1)} = S_r(q_r^{-1}) \tag{6.7}$$

for all $r \ge r_0$. On the other hand, consider the function S defined by

$$S(x) = \sum_{j=0}^{\infty}(j+1)a_j x^{j+1} = -xQ'(x). \tag{6.8}$$

Note that S is continuous on the interval $[0, R)$ and, hence, at $x=p=q^{-1}$ by our assumption. Thus, we have

$$\lim_{r\to\infty} S(q_r^{-1}) = S(q^{-1}) = \sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)} < +\infty. \tag{6.9}$$

Furthermore,

$$S_r(q_r^{-1}) = \sum_{j=0}^{r-1}(j+1)a_j q_r^{-(j+1)} \le S(q_r^{-1}) \tag{6.10}$$

for all $r \geq r_0$. Thus, by (6.7) and (6.10), we have $S_r(q^{-1}) \leq S(q_r^{-1})$ for all sufficiently large r and, hence, using (6.9) we see that $\lim_{r\to\infty} S_r(q_r^{-1}) = S(q^{-1}) < +\infty$. In other words, the denominator of (6.5) converges to that of (6.6) as r tends to ∞. Note that this value is not zero.

Let B_r denote the numerator of (6.5); i.e.,

$$B_r = \sum_{j=0}^{r-1}\left(\sum_{k=j}^{r-1} a_k q_r^{-(k+1)}\right) q_r^j \alpha_{-j} = \sum_{k=0}^{r-1} a_k q_r^{-(k+1)}\left(\sum_{j=0}^{k} q_r^j \alpha_{-j}\right).$$

Furthermore, set

$$C_r = \sum_{k=0}^{r-1} a_k q^{-(k+1)}\left(\sum_{j=0}^{k} q^j \alpha_{-j}\right) \quad \text{and} \quad H_r = \sum_{k=0}^{r-1} a_k q^{-(k+1)}\left(\sum_{j=0}^{k} q_r^j \alpha_{-j}\right)$$

so that we have

$$|B_r - C_r| \leq |B_r - H_r| + |H_r - C_r|. \tag{6.11}$$

First, let us consider $D_r = |B_r - H_r|$. We have

$$D_r \leq \sum_{k=0}^{r-1} a_k q_r^{-(k+1)}\left|1 - \frac{q^{-(k+1)}}{q_r^{-(k+1)}}\right|\left(\sum_{j=0}^{k} q_r^j |\alpha_{-j}|\right). \tag{6.12}$$

It is easy to see that $|1 - q^{-(k+1)}/q_r^{-(k+1)}| = |1 - (q_r/q)^{k+1}| \leq (k+1)(1 - (q_r/q))$ for all $k \geq 0$, since $q_r \leq q$. Thus, $D_r \leq (1 - q_r/q)\sum_{k=0}^{r-1}(k+1)a_k q_r^{-(k+1)}(\sum_{j=0}^{k} q_r^j |\alpha_{-j}|)$ by (6.12). Furthermore, since $q_r \leq q$, we have $q_r^j |\alpha_{-j}| \leq q^j |\alpha_{-j}| < K$ for all $j \geq 0$ by our assumption. Hence, we obtain $D_r \leq K(1 - q_r/q)\sum_{k=0}^{r-1}(k+1)^2 a_k q_r^{-(k+1)}$. Consider the function T defined by $T(x) = \sum_{j=0}^{\infty}(j+1)^2 a_j x^{j+1}$, which is continuous on the interval $[0, R)$, since $T(x) = xS'(x)$, where S is the function defined by (6.8). Since $0 < q^{-1} < R$ by our assumption and $\lim_{r\to\infty} q_r = q$, there exists an $r_2 \geq r_0$ such that $0 < q^{-1} \leq q_r^{-1} < R$ for all $r \geq r_2$. As $q_r \leq q_{r'}$ whenever $r < r'$, we obtain

$$D_r \leq K\left(1 - \frac{q_r}{q}\right)\sum_{k=0}^{r-1}(k+1)^2 a_k q_{r_2}^{-(k+1)} = KT(q_{r_2}^{-1})\left(1 - \frac{q_r}{q}\right) = M_1\left(1 - \frac{q_r}{q}\right) \tag{6.13}$$

for all $r \geq r_2$, where $M_1 = KT(q_{r_2}^{-1})$ is a positive constant.

For $E_r = |H_r - C_r|$, we have $E_r \leq \sum_{k=0}^{r-1} a_k q^{-(k+1)}(\sum_{j=0}^{k} |q_r^j - q^j||\alpha_{-j}|)$. Therefore,

$$\sum_{j=0}^{k} |q_r^j - q^j||\alpha_{-j}| = \sum_{j=0}^{k} q^j \left|1 - \left(\frac{q_r}{q}\right)^j\right| |\alpha_{-j}| \tag{6.14}$$

for every $k \geq 0$. Furthermore, since $0 < q_r \leq q$, we have $|1 - (q_r/q)^j| \leq j(1 - q_r/q)$. Hence, (6.1) together with (6.14) implies

$$\sum_{j=0}^{k} |q_r^j - q^j||\alpha_{-j}| \leq \left(1 - \frac{q_r}{q}\right)\sum_{j=0}^{k} j q^j |\alpha_{-j}| \leq \frac{K}{2}(k+1)^2\left(1 - \frac{q_r}{q}\right).$$

Then we have

$$E_r \leq \frac{K}{2}\left(1 - \frac{q_r}{q}\right)\sum_{k=0}^{\infty}(k+1)^2 a_k q^{-(k+1)} = M_2\left(1 - \frac{q_r}{q}\right), \tag{6.15}$$

where $M_2 = KT(q^{-1})/2$ is a positive constant.

By (6.11), (6.13), and (6.15), we have

$$|B_r - C_r| \le M\left(1 - \frac{q_r}{q}\right), \tag{6.16}$$

where $M = M_1 + M_2 > 0$. On the other hand, since

$$\sum_{k=0}^{r-1} a_k q^{-(k+1)}\left(\sum_{j=0}^{k} q^j |\alpha_{-j}|\right) \le K\sum_{k=0}^{r-1}(k+1)a_k q^{-(k+1)} \le KS(q^{-1}) < +\infty \tag{6.17}$$

by our assumptions, $\lim_{r\to\infty} C_r$ exists and is equal to

$$\sum_{k=0}^{\infty} a_k q^{-(k+1)}\left(\sum_{j=0}^{k} q^j \alpha_{-j}\right) = \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k q^{j-k-1}\right)\alpha_{-j}, \tag{6.18}$$

since (6.17) shows that the above series converges absolutely. Thus, by (6.16) together with the fact that $q = \lim_{r\to\infty} q_r$, we see that $\lim_{r\to\infty} B_r$ exists and is equal to the value as in (6.18), which is nothing but the numerator of (6.6). □

Lemma 6.6: Assume that one of the three conditions of Theorem 6.1 is satisfied. Then (6.1) implies condition (C_∞).

Proof: By (6.1), for all $n \ge 1$, we have

$$\sum_{j=0}^{\infty} a_{j+n-1}|\alpha_{-j}| \le K\sum_{j=0}^{\infty} a_{j+n-1}q^{-j} = Kq^{n-1}\sum_{j=0}^{\infty} a_{j+n-1}q^{-(j+n-1)} \le Kq^n,$$

since we have $\sum_{j=0}^{\infty} a_j q^{-(j+1)} = 1$. Thus, condition (C_∞) is satisfied. □

Combining Theorem 5.2, Lemma 6.5, and Lemma 6.6, we obtain Theorem 6.2.

When $p = R$, we have a partial result as follows.

Proposition 6.7: Assume that one of the three conditions of Theorem 6.1 is satisfied, that $d_{r_1} = 1$ for some $r_1 \ge r_0$, that $\sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)} = +\infty$, and that the series $\sum_{j=0}^{\infty} q^j |\alpha_{-j}|$ converges. If the sequences $\{V_n^{(r)}/q_r^n\}_{n\ge1}$ are uniformly convergent for $r \ge r_1$, then V_n exists for all n and we have $\lim_{n\to\infty} V_n/q^n = 0$.

Note that the above condition implies that $p = R$ [see (6.9)].

Proof of Proposition 6.7: Since we have $q \ge q_r$, we see easily that the numerator B_r of (6.5) satisfies

$$|B_r| \le \sum_{j=0}^{r-1} q_r^j |\alpha_{-j}| \le \sum_{j=0}^{r-1} q^j |\alpha_{-j}| \le \sum_{j=0}^{\infty} q^j |\alpha_{-j}| < +\infty. \tag{6.19}$$

The result now follows from Theorem 5.2, (6.5), Lemma 6.6, and (6.19). □

Remark 6.8: Results similar to Theorem 6.2 and Proposition 6.7 were obtained in Theorem 3.2 of [11] by using the Markov chain method. See, also, Theorem 3.10 of [8].

Problem 6.9: We do not know if $d_\infty = \gcd\{i+1 : a_i > 0\} = 1$ ($\Leftrightarrow d_{r_1} = 1$ for some $r_1 \ge r_0$) implies that $L = \lim_{n\to\infty} V_n/q^n$ exists in general. Note that in some special cases $d_\infty = 1$ if and only if $\lim_{n\to\infty} V_n/q^n$ exists, as was shown in [11].

7. EXAMPLE

Let us give an explicit example of our main theorem of the previous section.

Fix a real number $\alpha^{-1} = \beta > 1$ and set $\alpha_r^{-1} = \beta_r = \beta^{1-(1/r!)}$ for $r \geq 1$. Consider the sequence of real polynomials $\{U_r(x)\}_{r\geq 1}$ defined inductively by

$$U_1(x) = 2x - 2\beta_1, \tag{7.1}$$

$$U_{r+1}(x) = xU_r(x) - \beta_{r+1}U_r(\beta_{r+1}) \quad (r \geq 1). \tag{7.2}$$

Therefore, we have $U_r(x) = 2x^r - a_0x^{r-1} - \cdots - a_{r-2}x - a_{r-1}$ for some strictly positive real numbers a_j $(j \geq 0)$. Note that β_r is the unique positive real root of $U_r(x)$. Set $W_r(x) = 2 - a_0x - \cdots - a_{r-2}x^{r-1} - a_{r-1}x^r = x^rU_r(x^{-1})$. Then we have $W_r(0) = 2$ and $W_r(\alpha_r) = 0$. Furthermore, we set $W(x) = 2 - \sum_{j=0}^{\infty} a_jx^{j+1}$.

Lemma 7.1: We have $W(\alpha) = 0$ and $0 < \alpha \leq R$, where R is the radius of convergence of W.

Proof: Since $W_r(\alpha_r) = 0$ and $a_j = \beta_{j+1}U_j(\beta_{j+1}) \leq 2\beta_{j+1}^{j+1} \leq 2\beta^{j+1} = 2\alpha^{-(j+1)}$, we get $W_r(\alpha) = W_r(\alpha) - W_r(\alpha_r) = a_0(\alpha_r - \alpha) + a_1(\alpha_r^2 - \alpha^2) + \cdots + a_{r-1}(\alpha_r^r - \alpha^r)$. Thus,

$$\begin{aligned} W_r(\alpha) &\leq 2(\alpha_r - \alpha)/\alpha + 2(\alpha_r^2 - \alpha^2)/\alpha^2 + \cdots + 2(\alpha_r^r - \alpha^r)/\alpha^r \\ &= 2(\beta^{1/r!} - 1) + 2(\beta^{2/r!} - 1) + \cdots + 2(\beta^{r/r!} - 1). \end{aligned}$$

Therefore, we have

$$W_r(\alpha) \leq 2r(\beta^{1/(r-1)!} - 1) = (2r/(r-1)!)(r-1)!(\beta^{1/(r-1)!} - 1) \to 0 \quad (r \to \infty).$$

Thus, $W(\alpha) = \lim_{r\to\infty} W_r(\alpha) = 0$. □

Set $Q_r(x) = W_r(x) - 1$ and $Q(x) = W(x) - 1$. Then, for each $r \geq 1$, there exists a unique positive real root p_r of Q_r. Furthermore, by Theorem 6.1, $p = \lim_{r\to\infty} p_r$ exists and $Q(p) = 0$. Set $q_r = p_r^{-1}$ and $q = p^{-1}$ and note that $0 < p < R$, where R coincides with the radius of convergence of Q.

Lemma 7.2:

$$\lim_{r\to\infty} \left| \frac{p_r^r}{p^r} - 1 \right| = 0. \tag{7.3}$$

Proof: Let us fix an $r \geq 1$ for the moment. The functions $W(x)$ and $W_r(x)$ defined on the intervals $[0, d)$ and $[0, \infty)$, respectively, are differentiable with strictly negative derivatives. Let us denote by $g: (0, 2] \to [0, d)$ and $g_r: (-\infty, 2] \to [0, \infty)$, respectively, their inverse functions. Then define the differentiable function $f: (0, 2] \to \mathbf{R}$ by $f(y) = g(y)^r - g_r(y)^r$. For $y \in (0, 2)$, set $x = g(y)$ and $x_r = g_r(y)$. Then we obtain $x_r \geq x > 0$ and

$$\begin{aligned} -\frac{W'(x)}{x^{r-1}} &= \frac{a_0}{x^{r-1}} + 2\frac{a_1}{x^{r-2}} + \cdots + (r-1)\frac{a_{r-2}}{x} + ra_{r-1} + (r+1)a_rx + \cdots \\ &\geq \frac{a_0}{x_r^{r-1}} + 2\frac{a_1}{x_r^{r-2}} + \cdots + (r-1)\frac{a_{r-2}}{x_r} + ra_{r-1} = -\frac{W_r'(x_r)}{x_r^{r-1}} > 0. \end{aligned} \tag{7.4}$$

Hence, by (7.4), we have $f'(y) = rx^{r-1}W'(x)^{-1} - rx_r^{r-1}W_r'(x)^{-1} \geq 0$. Thus, the function f is nondecreasing and we obtain $\alpha^r - \alpha_r^r = \lim_{y\to+0} f(y) \leq f(1) = p^r - p_r^r$. Therefore,

$$|p^r - p_r^r| = p_r^r - p^r \le |\alpha^r - \alpha_r^r|$$

for all $r \ge 1$. Then we have

$$\left|\frac{p_r^r}{p^r} - 1\right| \le \left(\frac{\alpha}{p}\right)^r \left|\frac{\alpha_r^r}{\alpha^r} - 1\right| = \left(\frac{\alpha}{p}\right)^r |\beta^{1/(r-1)!} - 1| = \left(\frac{\alpha}{p}\right)^r \frac{1}{(r-1)!} \frac{|\beta^{1/(r-1)!} - 1|}{1/(r-1)!}. \tag{7.5}$$

Since $\lim_{r\to\infty}(\alpha / p)^r / (r-1)! = 0$ and $\lim_{r\to\infty}|\beta^{1/(r-1)!} - 1|(r-1)! = \ln\beta$, equation (7.3) holds. □

Let $\{V_n\}_{n\in Z}$ be the ∞-GFS defined by $V_n = q^n$. Let us show that the conditions of Theorem 6.2 are satisfied for this sequence. Recall that we denoted $x_n^{(r)} = V_n^{(r)}/q_r^n$; see Theorem 5.2.

Lemma 7.3: The sequences $\{x_n^{(r)}\}_{n\ge1}$ are uniformly convergent for $r \ge 1$.

Proof: By Lemma 7.2, for a given $\varepsilon > 0$, there exists an $r_2 > 0$ such that $|p^r / p_r^r - 1| < \varepsilon / 2$ for all $r \ge r_2$. Let us fix an r with $r \ge r_2$. Then, by (3.1), for every n with $-r+1 \le n \le 0$, we have

$$|x_n^{(r)} - 1| = \left|\frac{V_n^{(r)}}{q_r^n} - 1\right| = \left|\frac{q^n}{q_r^n} - 1\right| \le \left|\left(\frac{q}{q_r}\right)^{-r} - 1\right| = \left|\frac{p^r}{p_r^r} - 1\right| < \frac{\varepsilon}{2}. \tag{7.6}$$

Suppose $|x_k^{(r)} - 1| < \varepsilon / 2$ for all k with $-r+1 \le k \le n$, where $n \ge 0$. Then, by (6.4) and the relation $x_{n+1}^{(r)} = (a_0 / q_r)x_n^{(r)} + (a_1 / q_r^2)x_{n-1}^{(r)} + \cdots + (a_{r-1} / q_r^r)x_{n-r+1}^{(r)}$, we have

$$|x_{n+1}^{(r)} - 1| = \left|\frac{a_0}{q_r}(x_n^{(r)} - 1)\right| + \left|\frac{a_1}{q_r^2}(x_{n-1}^{(r)} - 1)\right| + \cdots + \left|\frac{a_{r-1}}{q_r^r}(x_{n-r+1}^{(r)} - 1)\right| < \frac{\varepsilon}{2}. \tag{7.7}$$

Thus, by induction, we see that $|x_n^{(r)} - 1| < \varepsilon / 2$ for all n, provided that $r \ge r_2$.

On the other hand, by Lemma 6.4, $L_r = \lim_{n\to\infty} x_n^{(r)}$ exists for all $r \ge 1$ and we can check that $\lim_{r\to\infty} L_r = 1$ by using (6.5). Hence, there exists an $r_3 \ge r_2$ such that $|L_r - 1| < \varepsilon / 2$ for all $r \ge r_3$. Therefore, for all $r \ge r_3$ and all $n \ge 1$, we have $|x_n^{(r)} - L_r| \le |x_n^{(r)} - 1| + |1 - L_r| < \varepsilon / 2 + \varepsilon / 2 = \varepsilon$. Since we have only a finite number of r's with $r_3 > r \ge 1$, there exists an N such that $|x_n^{(r)} - L_r| < \varepsilon$ for all $n \ge N$ and all r with $r_2 > r \ge 1$. Thus, we have proved that the sequences $\{x^{(r)}\}_{n\ge1}$ are uniformly convergent for $r \ge 1$. □

Therefore, we have shown that all the conditions in Theorem 6.2 are satisfied. On the other hand, we see easily that

$$\lim_{n\to\infty} \frac{V_n}{q^n} = \frac{\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} a_k q^{j-k-1}\right) q^{-j}}{\sum_{j=0}^{\infty}(j+1)a_j q^{-(j+1)}} = 1. \tag{7.8}$$

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