MORGAN-VOYCE POLYNOMIAL DERIVATIVE SEQUENCES

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1. INTRODUCTION

The aim of this paper is to study the main properties of the derivatives $B_n^{(1)}(x)$ and $C_n^{(1)}(x)$ of the *Morgan-Voyce polynomials* $B_n(x)$ and $C_n(x)$ (e.g., see [8]) described in the next section. Here x is an indeterminate and the bracketed superscript symbolizes the first derivative with respect to x. As done in previous papers, we shall confine ourselves to considering the case $x = 1$. For notational convenience, the terms $B_n^{(1)}(1)$ and $C_n^{(1)}(1)$ will be denoted by R_n and S_n, respectively.

Various papers have dealt with this kind of polynomial pairs. For example, the polynomial pairs (Fibonacci, Lucas), (Pell, Pell-Lucas), and (Jacobsthal, Jacobsthal-Lucas) have been studied in [1], [2], [3], [4], [9], and [10].

The paper is set out as follows. After recalling some background material on the Morgan-Voyce polynomials, we show first some basic properties of the numbers R_n and S_n the most interesting of which are, perhaps, expressions for sums and differences involving subscript sums and differences (see Section 3.3). In Section 4, we evaluate certain finite sums involving R_n and S_n. We conclude the paper with some properties of divisibility and the primality of R_n and S_n.

1.1 Some Useful Results for Fibonacci and Lucas Numbers F_n, L_n

Binet forms are

$$F_n = (a^n - b^n)/\sqrt{5}, \tag{1.1}$$

$$L_n = a^n + b^n, \tag{1.2}$$

where a and b are the roots of the equation $t^2 - t - 1 = 0$, i.e.,

$$a = (1+\sqrt{5})/2,\ b = (1-\sqrt{5})/2 \ \ (\text{so } a+b=1,\ ab=-1,\ a-b=\sqrt{5}). \tag{1.3}$$

From (1.1)-(1.3), it follows readily that

$$F_{n+2p} + F_{n-2p} = F_n L_{2p}, \tag{1.4}$$

$$F_{n+2p} - F_{n-2p} = L_n F_{2p}, \tag{1.5}$$

$$L_{n+2p} + L_{n-2p} = L_n L_{2p}, \tag{1.6}$$

$$L_{n+2p} - L_{n-2p} = 5F_n F_{2p}. \tag{1.7}$$

Some relationships among Fibonacci, Lucas, and Morgan-Voyce polynomials that are applicable to the development of our theme include

$$xB_n(x^2) = F_{2n}(x), \tag{1.8}$$

$$C_n(x^2) = L_{2n}(x). \tag{1.9}$$

These occur as (4.1) and (4.3) of [8]. Substituting $x = 1$ in this pair of relationships yields

$$B_n = F_{2n}, \tag{1.8a}$$

$$C_n = L_{2n}, \tag{1.9a}$$

where $B_n := B_n(1), \ldots$.

Background information on the Fibonacci and Lucas numbers may be found in [11].

2. BACKGROUND MATERIAL

Consider the polynomial sequence $\{X_n(x)\}$ defined by the recurrence

$$X_n(x) = (x+2)X_{n-1}(x) - X_{n-2}(x) \quad (n \geq 2) \tag{2.1}$$

with initial conditions

$$X_0(x) = a_0, \;\; X_1(x) = a_1 \quad (a_0, a_1 \text{ integers}). \tag{2.2}$$

Special cases for the Morgan-Voyce polynomials $B_n(x)$ and $C_n(x)$ are:

$$\begin{cases} (a_0, a_1) = (0, 1) & \text{if } X_n(x) \equiv B_n(x), \\ (a_0, a_1) = (2, 2+x) & \text{if } X_n(x) \equiv C_n(x). \end{cases} \tag{2.3}$$

It has to be pointed out that, in the very special case $x = 0$, we have

$$B_n(0) = n \quad \text{and} \quad C_n(0) = 2 \;\forall n. \tag{2.4}$$

Combinatorial expressions for the above polynomials are

$$B_n(x) = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} x^k \quad (n \geq 0) \quad [8, (2.20)], \tag{2.5}$$

$$C_n(x) = \sum_{k=0}^{n-1} \frac{2n}{n-k} \binom{n+k-1}{2k} x^k + x^n \quad (n \geq 1) \quad [8, (3.22)]. \tag{2.6}$$

Observe that, if we assume that $0^0 = 1$ (see [5] for some considerations on this assumption), then (2.5) and (2.6) hold also for $x = 0$ [cf. (2.4)].

Binet forms are

$$B_n(x) = (\alpha^n - \beta^n) / \Delta, \tag{2.7}$$

$$C_n(x) = \alpha^n + \beta^n, \tag{2.8}$$

where the roots $\alpha := \alpha(x)$, $\beta := \beta(x)$ of the characteristic equation $t^2 - (x+2)t + 1 = 0$ are

$$\alpha = (x+2+\Delta)/2, \;\; \beta = (x+2-\Delta)/2 \tag{2.9}$$

so that

$$\alpha + \beta = x + 2, \;\; \alpha\beta = 1, \;\; \alpha - \beta = \Delta := \Delta(x) = \sqrt{x(x+4)}. \tag{2.10}$$

Clearly, (1.3) contrasted with (2.9) and (2.10) together reveals that $a^2 = \alpha(1)$, $b^2 = \beta(1)$.

Notice that

$$\alpha^{(1)} := \frac{d\alpha(x)}{dx} = \frac{\alpha}{\Delta}, \;\; \beta^{(1)} := \frac{d\beta(x)}{dx} = -\frac{\beta}{\Delta}, \tag{2.11}$$

leading to

$$(\alpha^n)^{(1)} = \frac{n\alpha^n}{\Delta}, \; (\beta^n)^{(1)} = -\frac{n\beta^n}{\Delta}, \tag{2.12}$$

and

$$\Delta^{(1)} := \frac{d\Delta(x)}{dx} = \frac{x+2}{\Delta(x)}. \tag{2.13}$$

3. SOME ELEMENTARY PROPERTIES OF R_n AND S_n

3.1 Basics

From (2.5) and (2.6) we immediately obtain the derivatives

$$B_n^{(1)}(x) := \frac{d}{dx}B_n(x) = \sum_{k=0}^{n-1} k\binom{n+k}{2k+1}x^{k-1} \quad (n \geq 0), \tag{3.1}$$

$$C_n^{(1)}(x) := \frac{d}{dx}C_n(x) = \sum_{k=0}^{n-1} \frac{2nk}{n-k}\binom{n+k-1}{2k}x^{k-1} + nx^{n-1} \quad (n \geq 1). \tag{3.2}$$

For example, $B_4^{(1)}(x) = 3x^2 + 12x + 10$ and $C_4^{(1)}(x) = 4x^3 + 24x^2 + 40x + 16$.

When $x = 1$, the following table can be constructed $[B_n^{(1)}(1) := R_n, \; C_n^{(1)}(1) := S_n]$ from (3.1) and (3.2).

TABLE 1. Values of R_n and S_n for $0 \leq n \leq 10$

n	0	1	2	3	4	5	6	7	8	9	10
R_n	0	0	1	6	25	90	300	954	2939	8850	26195
S_n	0	1	6	24	84	275	864	2639	7896	23256	67650

Observe that the value of R_0 can be obtained by letting $x = 1$ in (3.1) with the assumption that a sum vanishes whenever the upper range indicator is less than the lower one. The value of S_0 comes from the fact that the initial condition $C_0(x) = 2$ is independent of x.

Using the Binet forms (2.7) and (2.8) with (2.12) and (2.13), we deduce that

$$B_n^{(1)}(x) = [nC_n(x) - (x+2)B_n(x)] / \Delta^2, \tag{3.3}$$

$$C_n^{(1)}(x) = nB_n(x) \quad \text{as in [8, (3.24)]}, \tag{3.4}$$

whence

$$R_n := B_n^{(1)}(1) = (nL_{2n} - 3F_{2n}) / 5 \quad \text{[by (1.8a) and (1.9a)]}, \tag{3.5}$$

$$S_n := C_n(1) = nF_{2n} \quad \text{[by (1.10a)]}, \tag{3.6}$$

results which are of subsequent application.

3.2 Negative Subscripts

Direct differentiation of $B_{-n}(x) = -B_n(x)$, $C_{-n}(x) = C_n(x)$ [8, (5.1), (5.2)] yields

$$R_{-n} = -R_n, \tag{3.7}$$

$$S_{-n} = S_n. \tag{3.8}$$

3.3 Sums and Differences Involving Subscript Sums and Differences

Routine algebraic computation applied to standard Fibonacci and Lucas number knowledge [see (1.4)-(1.7)] with (3.5) produces the identities

$$R_{n+p}+R_{n-p}=L_{2p}R_n+F_{2n}S_p, \tag{3.9}$$

$$R_{n+p}-R_{n-p}=L_{2p}R_p+F_{2p}S_n, \tag{3.10}$$

with special cases

($p=1$):
$$R_{n+1}+R_{n-1}=3R_n+F_{2n}, \tag{3.11}$$

$$R_{n+1}-R_{n-1}=nF_{2n}=S_n \quad \text{by (3.6)}. \tag{3.12}$$

($p=n$):
$$R_{2n}=L_{2n}R_n+nF_{2n}^2=L_{2n}R_n+F_{2n}S_n. \tag{3.13}$$

Furthermore, with (3.6),

$$S_{n+p}+S_{n-p}=L_{2p}S_n+L_{2n}S_p, \tag{3.14}$$

$$S_{n+p}-S_{n-p}=nL_{2n}F_{2p}+pF_{2n}L_{2p}, \tag{3.15}$$

whence

$$S_{2n}=2L_{2n}S_n. \tag{3.16}$$

4. EVALUATION OF SOME FINITE SUMS FOR R_n AND S_n

As a calculational aid in the ensuing investigations, we need the following identities [1], [7] which are valid for arbitrary y:

$$\sum_{r=0}^{k} ry^r=[ky^{k+2}-(k+1)y^{k+1}+y]/(y-1)^2, \tag{4.1}$$

$$\sum_{r=0}^{k}\binom{k}{r}y^r=(y+1)^k, \tag{4.2}$$

$$\sum_{r=0}^{k}\binom{k}{r}ry^r=ky(y+1)^{k-1}. \tag{4.3}$$

Proposition 1:

$$\sum_{r=0}^{k} R_r=1+(kL_{2k+1}-L_{2k}-3F_{2k+1})/5. \tag{4.4}$$

Proposition 2:

$$\sum_{r=0}^{k} S_r=kF_{2k+1}-F_{2k}. \tag{4.5}$$

Proof of Proposition 1: Taking (3.5) into account, rewrite the left-hand side of (4.4) as

$$\begin{aligned}
&\frac{1}{5}\left[\sum_{r=0}^{k} rL_{2r}-3\sum_{r=0}^{k} F_{2r}\right]\\
&=\frac{1}{5}\left[\sum_{r=0}^{k} r(a^{2r}+b^{2r})-\frac{3}{\sqrt{5}}\sum_{r=0}^{k}(a^{2r}-b^{2r})\right] \quad \text{by (1.1), (1.2)}\\
&=[kL_{2k+2}-(k+1)L_{2k}+2-3(F_{2k+1}-1)]/5 \quad \text{by (4.1) with } y=a^2, b^2\\
&=(kL_{2k+1}-L_{2k}-3F_{2k+1}+5)/5. \ \square
\end{aligned}$$

Identity (4.5) can be proved in a way similar to that for (4.4).

Proposition 3:

$$\sum_{r=0}^{k}\binom{k}{r}R_r=\begin{cases}5^{(k-2)/2}(kF_{k+1}-3F_k) & (k \text{ even}),\\ 5^{(k-3)/2}(kL_{k+1}-3L_k) & (k \text{ odd}).\end{cases} \tag{4.6}$$

Proposition 4:

$$\sum_{r=0}^{k}\binom{k}{r}S_r=\begin{cases}5^{(k-2)/2}kL_{k+1} & (k \text{ even}),\\ 5^{(k-1)/2}kF_{k+1} & (k \text{ odd}).\end{cases} \tag{4.7}$$

To prove Propositions 3 and 4, we need the following lemmas.

Lemma 1:

$$\sum_{r=0}^{k}\binom{k}{r}F_{2r}=\begin{cases}5^{k/2}F_k & (k \text{ even}),\\ 5^{(k-1)/2}L_k & (k \text{ odd}).\end{cases} \tag{4.8}$$

Lemma 2:

$$\sum_{r=0}^{k}\binom{k}{r}rL_{2r+m}=\begin{cases}5^{k/2}kF_{k+m+1} & (k \text{ even}),\\ 5^{(k-1)/2}kL_{k+m+1} & (k \text{ odd}).\end{cases} \tag{4.9}$$

Lemma 3:

$$\sum_{r=0}^{k}\binom{k}{r}rF_{2r+m}=\begin{cases}5^{(k-2)/2}kL_{k+m+1} & (k \text{ even}),\\ 5^{(k-1)/2}kF_{k+m+1} & (k \text{ odd}).\end{cases} \tag{4.10}$$

To prove these three lemmas, use (1.1)-(1.2) along with (4.2)-(4.3) while recalling the key relationships $a^2+1=a\sqrt{5}$ and $b^2+1=-b\sqrt{5}$ deduced from (1.3).

Proof of Proposition 3 (a sketch): From (3.5), rewrite the left-hand side of (4.6) as

$$\frac{1}{5}\sum_{r=0}^{k}\binom{k}{r}(rL_{2r}-3F_{2r}),$$

whence the right-hand side of (4.6) can be obtained after some algebraic enterprises involving (4.8), and (4.9) with $m=0$. □

With the aid of Lemma 3, Proposition 4 can be proved in a similar way.

5. SOME DIVISIBILITY PROPERTIES OF R_n AND S_n

In this section, the divisibility of R_n and S_n by the first three primes is investigated. To save space, only Proposition 7 is proved in detail. A glimpse to the primality of the integers under study is caught at the end of the section.

Proposition 5: ***(i)*** R_n is odd iff $n=2(3k\pm1)$, while ***(ii)*** S_n is odd iff $n=6k\pm1$.

Proposition 6: ***(i)*** R_n is divisible by 3 iff either $n=3k$ or $n=6k\pm1$, while ***(ii)*** S_n is divisible by 3 iff either $n=2k$ or $n=3k$.

Corollary to Propositions 5 and 6: Both R_n and S_n are divisible by 3 iff they are even.

Proposition 7: ***(i)*** R_n is divisible by 5 iff either $n=5k$ or $n=5k\pm1$, while ***(ii)*** S_n is divisible by 5 iff $n=5k$.

Proof: The proof of (ii) is trivial as it is based on (3.6) and the well-known fact that F_n is divisible by 5 iff n is. As for (i), from (ii) and (3.12) we can say that

$$R_{n+1} \equiv R_{n-1} \pmod 5 \Leftrightarrow n \equiv 0 \pmod 5. \tag{5.1}$$

Further, from the recurrence $R_n = 3R_{n-1} - R_{n-2} + F_{2n-2}$ [that is readily obtained by calculating at $x = 1$ the first derivative with respect to x of both sides of (2.1) with $X \equiv B$, and using (1.8a)], and from the conditions on n for F_n to be divisible by 5, we have that

$$R_{n+1} \equiv 3R_n - R_{n-1} \pmod 5 \Leftrightarrow n \equiv 0 \pmod 5. \tag{5.2}$$

From (5.1)-(5.2) we can write $2R_{n+1} \equiv 3R_n \pmod 5 \Rightarrow n \equiv 0 \pmod 5$, that is,

$$R_{n+1} \equiv -R_n \pmod 5 \Rightarrow n \equiv 0 \pmod 5. \tag{5.3}$$

From (5.3) and (5.1), it remains to prove that

$$n \equiv 0 \pmod 5 \Rightarrow R_n \equiv 0 \pmod 5. \tag{5.4}$$

Put $n = 5k$ in (3.5) thus getting

$$R_{5k} = kL_{10k} - 3F_{10k}/5. \tag{5.5}$$

On using (2.4)-(2.4′) of [6], we can express $F_{10k}/5$ (for k even) as

$$\sum_{r=1}^{k/2}(F_{20r-17} + F_{20r-14} + F_{20r-4} + F_{20r-7} + F_{20r-9} + F_{20r-11}) \tag{5.6}$$

and (for k odd)

$$11 + \sum_{r=1}^{(k-1)/2}(F_{20r-7} + F_{20r-4} + F_{20r+6} + F_{20r+3} + F_{20r+1} + F_{20r-1}). \tag{5.7}$$

For $r = 1$, expression (5.6) is congruent to 3 modulo 5. Since the repetition period of the Fibonacci sequence reduced modulo 5 is 20, the congruence above holds for all $r \le k/2$. It follows that $F_{10k}/5 \equiv 3k/2 \pmod 5$ if k is even. Analogously, it can be seen from (5.7) that $F_{10k}/5 \equiv k \pmod 5$ if k is odd. Summarizing, we found that

$$3F_{10k}/5 \equiv \begin{cases} 2k \pmod 5 & (k \text{ even}), \\ 3k \pmod 5 & (k \text{ odd}). \end{cases} \tag{5.8}$$

Finally, the inspection of the sequence $\{kL_{10k}\}$ reduced modulo 5 shows that

$$kL_{10k} \equiv \begin{cases} 2k \pmod 5 & (k \text{ even}), \\ 3k \pmod 5 & (k \text{ odd}). \end{cases} \tag{5.9}$$

Identity (5.5) along with congruences (5.8) and (5.9) prove (5.4) and the proposition. □

5.1 On the Primality of R_n and S_n

Since $S_n \equiv 0 \pmod n$ for $n \ge 1$ [see (3.6)], these integers cannot be prime. From Propositions 5-7, we see that a necessary condition for R_n to be a prime is that $n \equiv 2$, or 8, or 22, or 28 (mod 60). By using the function "nextprime" of the software package DERIVE, we found only two prime R_n for $n \le 248$, namely,

$$R_8 = 2939 \quad \text{and} \quad R_{68} = 352{,}536{,}175{,}722{,}757{,}107{,}150{,}131{,}558{,}879.$$

6. CONCLUSIONS

What has been presented in the preceding theory provides us with some feeling for the flow of ideas emanating from the initial sources.

Future directions of related research studies could lead to the investigation of partial derivative aspects of the Morgan-Voyce polynomials and, perhaps more importantly, to the *integration sequences* associated with these mathematically fertile polynomials.

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