DIOPHANTINE TRIPLETS AND THE PELL SEQUENCE

M. N. Deshpande
Institute of Science, Nagpur 440 001, India

Ezra Brown
Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123
(Submitted April 1999-Final Revision August 2000)

1. INTRODUCTION

A Diophantine triplet is a set of three positive integers (a, b, c) such that $a < b < c$ and $ab = 1$, $bc + 1$, and $ac + 1$ are integer squares. Examples of such triplets are $(1, 3, 8)$, $(2, 4, 12)$, and $(2990, 22428, 41796)$.

The following four families of Diophantine triplets are well known:

$$\begin{aligned}
\mathbf{F}_1 &= \{(F_{2n}, F_{2n+2}, F_{2n+4}) : n \geq 1\},\\
\mathbf{F}_2 &= \{(F_{2n}, F_{2n+4}, 5F_{2n+2}) : n \geq 1\},\\
\mathbf{P}_1 &= \{(P_{2n}, 2P_{2n}, P_{2n+2}) : n \geq 1\},\\
\mathbf{P}_2 &= \{(P_{2n}, P_{2n+2}, 2P_{2n+2}) : n \geq 1\}.
\end{aligned}$$

We refer readers to [2], [3], and [4] for these families. Here, F_k and P_k denote the k^{th} element of the Fibonacci sequence and the Pell sequence, respectively. In [1], the first author posed the problem of finding infinitely many such Diophantine triplets.

The aim of this paper is to construct several different infinite families of Diophantine triplets using elements of the Pell sequence. We then formulate and prove a general result which gives formulas for a doubly infinite family of Diophantine triples. We conclude with a result on Diophantine quadruplets.

2. THE PELL SEQUENCE

Although the Pell sequence is quite well known, we describe it here for the sake of completeness. The Pell sequence is the sequence $\{P_n\}$, where $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 1$. That is, the Pell sequence is the sequence $\{1, 2, 5, 12, 29, 70, 169, 408, \ldots\}$. (We note that this is the sequence of denominators for the successive convergents to the continued fraction expansion of $\sqrt{2}$.) The following two properties of the Pell sequence are used in this paper:

Property 1: P_n is even if n is even.

Property 2: For all $n \geq 1$, $2P_{2n}^2 + 1 = \left(\frac{3P_{2n} - P_{2n-2}}{2}\right)^2$.

3. SOME FAMILIES OF DIOPHANTINE TRIPLETS

For convenience, we shall use the following notation. $FP(k)$ denotes the k^{th} family obtained by using the Pell sequence. The n^{th} element of $FP(k)$ is a triple denoted by $T_n(k)$, whose elements in turn are denoted by $A_{n,k}$, $B_{n,k}$, and $C_{n,k}$. That is,

$$FP(k) = \{T_n(k) : n = 1, 2, \ldots\}, \text{ and } T_n(k) = (A_{n,k}, B_{n,k}, C_{n,k}).$$

Theorem 1: Let $A_{n,1}=\frac{P_{2n}}{2}$, $B_{n,1}=4P_{2n}$, and $C_{n,1}=\frac{15}{2}P_{2n}-P_{2n-2}$. Then

$$FP(1)=\{T_n(1):n=1,2,\ldots\}=\{(A_{n,1},B_{n,1},C_{n,1}):n=1,2,\ldots\}$$

is a family of Diophantine triplets.

Proof: For each $n\geq 1$, using the definition of the Pell sequence and Property 2 leads to the equations

$$A_{n,1}B_{n,1}+1=2P_{2n}^2+1=\left(\frac{3P_{2n}-P_{2n-2}}{2}\right)^2,$$

$$\begin{aligned}A_{n,1}C_{n,1}+1&=\frac{P_{2n}}{2}\left(\frac{15}{2}P_{2n}-P_{2n-2}\right)+1\\&=\frac{P_{2n}}{2}\left(\frac{15}{2}P_{2n}-P_{2n-2}\right)+\left(\frac{3P_{2n}-P_{2n-2}}{2}\right)^2-2P_{2n}^2\\&=4P_{2n}^2-2P_{2n}P_{2n-2}+\frac{P_{2n-2}^2}{4}=\left(2P_{2n}-\frac{P_{2n-2}}{2}\right)^2,\end{aligned}$$

and

$$\begin{aligned}B_{n,1}C_{n,1}+1&=4P_{2n}\left(\frac{15}{2}P_{2n}-P_{2n-2}\right)+1\\&=4P_{2n}\left(\frac{15}{2}P_{2n}-P_{2n-2}\right)+\left(\frac{3P_{2n}-P_{2n-2}}{2}\right)^2-2P_{2n}^2\\&=\frac{121}{4}P_{2n}^2-\frac{11}{2}P_{2n}P_{2n-2}+P_{2n-2}^2\\&=\left(\frac{11}{2}P_{2n}-\frac{P_{2n-2}}{2}\right)^2.\end{aligned}$$

By Property 1, since P_{2n} and P_{2n-2} are even, each of the above squared expressions is an integer square, and the result follows.

We list a few elements of $FP(1)$:

n	Triple
1	$T_1(1)=(1,8,15)$
2	$T_2(1)=(6,48,88)$
3	$T_3(1)=(35,280,513)$
4	$T_4(1)=(204,1632,2990)$

Theorem 2: Let $A_{n,2}=2P_{2n}$, $B_{n,2}=15P_{2n}-2P_{2n-2}$, and $C_{n,2}=28P_{2n}-3P_{2n-2}$. Then $FP(2)=\{T_n(2):n=1,2,\ldots\}=\{(A_{n,2},B_{n,2},C_{n,2}):n=1,2,\ldots\}$ is a family of Diophantine triplets.

Proof: Noting that $A_{n,2}B_{n,2}=B_{n,1}C_{n,1}$, we see from Theorem 1 that

$$A_{n,2}B_{n,2}+1=\left(\frac{11}{2}P_{2n}-\frac{P_{2n-2}}{2}\right)^2.$$

Using algebraic techniques similar to those used in the proof of Theorem 1, we also obtain

$$A_{n,2}C_{n,2}+1=\left(\frac{15}{2}P_{2n}-P_{2n-2}\right)^2 \quad\text{and}\quad B_{n,2}C_{n,2}+1=\left(\frac{41}{2}P_{2n}-\frac{5}{2}P_{2n-2}\right)^2.$$

Again, since the subscripts are even, these squares are integer squares, and the result follows.

A few triples in the family $FP(2)$ are listed here:

n	Triple
1	$T_1(2)=(4,30,56)$
2	$T_2(2)=(24,176,330)$
3	$T_3(2)=(140,1026,1924)$
4	$T_4(2)=(816,5980,11214)$

Theorem 3: Let $A_{n,3}=\frac{15}{2}P_{2n}-P_{2n-2}$, $B_{n,3}=56P_{2n}-6P_{2n-2}$, and $C_{n,3}=\frac{209}{2}P_{2n}-12P_{2n-2}$. Then $FP(3)=\{T_n(3):n=1,2,\dots\}=\{(A_{n,3},B_{n,3},C_{n,3}):n=1,2,\dots\}$ is a family of Diophantine triplets.

Proof: Noting that $A_{n,3}B_{n,3}=B_{n,2}C_{n,2}$, we see from Theorem 2 that

$$A_{n,3}B_{n,3}+1=\left(\frac{41}{2}P_{2n}-\frac{5}{2}P_{2n-2}\right)^2.$$

Using algebraic techniques similar to those used in the proof of Theorem 1, we also obtain

$$A_{n,3}C_{n,3}+1=\left(28P_{2n}-\frac{7}{2}P_{2n-2}\right)^2 \quad\text{and}\quad B_{n,3}C_{n,3}+1=\left(\frac{153}{2}P_{2n}-\frac{17}{2}P_{2n-2}\right)^2.$$

As before, since the subscripts are even, these squares are integer squares, and the result follows.

Here are a few triples in the family $FP(3)$:

n	Triple
1	$T_1(3)=(15,112,209)$
2	$T_2(3)=(88,660,1230)$
3	$T_3(3)=(513,3848,7171)$
4	$T_4(3)=(2990,22428,41796)$

4. A DOUBLY INFINITE FAMILY OF DIOPHANTINE TRIPLETS

It is apparent from the previous section that the families $FP(1)$, $FP(2)$, and $FP(3)$ fit into an infinite family of such families. In this section we will derive formulas for such a double infinite family.

First, we define the auxiliary sequences $\{G_n:n\ge 1\}$, $\{H_n:n\ge 1\}$, and $\{S_n:n\ge 1\}$ by

$$\begin{aligned}
&G_1=1, G_2=4, \text{ and } G_{n+2}=4G_{n+1}-G_n && \text{for } n\ge 1;\\
&H_1=H_2=0, \text{ and } H_{n+2}=4H_{n+1}-H_n-2(-1)^n && \text{for } n\ge 1;\\
&S_1=4, S_2=14, \text{ and } S_{n+2}=4S_{n+1}-S_n && \text{for } n\ge 1.
\end{aligned}$$

Thus, $(G_n) = (1, 4, 15, 56, 209, 780, 2911, \dots)$, $(H_n) = (0, 0, 2, 6, 24, 88, 330, 1230, \dots)$, and $(S_n) = (4, 14, 52, 194, 724, \dots)$.

Our main result is the following.

Theorem 4: Let n and k be positive integers, and let

$$E(n,k) = \frac{G_k P_{2n} - H_k P_{2n-2}}{2}.$$

Then $(E(n,k), 2E(n,k+1), E(n,k+2))$ is a Diophantine triplet.

If we now define $FP(k) = \{(E(n,k), 2E(n,k+1), E(n,k+2)) : n = 1, 2, \dots\}$, then the cases $k = 1, 2$, and 3 agree with our previous definitions. Hence, this is the doubly infinite family we seek.

Proof of Theorem 4 uses the properties of the G_n, H_n, and S_n contained in Propositions G, H, and S; their proofs use only induction and some tedious but straightforward algebra.

Define the algebraic integers γ and δ by $\gamma = 2 + \sqrt{3}$ and $\delta = 2 - \sqrt{3}$.

Proposition G: For n a positive integer:

(1) $G_n = \dfrac{\gamma^n - \delta^n}{\gamma - \delta}$,

(2) $G_{n+2}G_n + 1 = G_{n+1}^2$,

(3) $2G_nG_{n+1} + 1 = (G_{n+1} - G_n)^2$,

(4) $G_{n+3} + G_n = 3(G_{n+1} + G_{n+2})$.

Proposition H: For n a positive integer:

(1) $H_n = \dfrac{(1+\sqrt{3})\gamma^{n-2} - (\sqrt{3}-1)\delta^{n-2}}{6} - \dfrac{(-1)^{n-2}}{3}$,

(2) $H_{n+2}H_n + 1 = (H_{n+1} + (-1)^{n+1})^2$,

(3) $H_n + H_{n+1} = 2G_{n-1}$,

(4) $H_{n+3} + H_n = 3(H_{n+1} + H_{n+2})$.

Proposition S: For n a positive integer:

(1) $S_n = \gamma^n + \delta^n$,

(2) $S_n + (-1)^n$ is divisible by 3,

(3) $S_{n+3} + S_n = 3(S_{n+1} + S_{n+2})$,

(4) $G_{n+2} - G_n = S_{n+1}$.

Remark: The reader will note that (G_n) and (S_n) are related in the same way as the Fibonacci and Lucas sequences are related.

The following lemmas are quite useful in deriving the main result. We give the proof of Lemma 1 here; proofs of Lemmas 2 and 3 are similar but longer, and we have relegated them to the appendix.

Lemma 1: For every positive integer n, $2H_nH_{n+1} + 1 = \left(\frac{S_{n-1} + (-1)^{n-1}}{3}\right)^2$.

Lemma 2: For every positive integer n, $G_nH_{n+2}+G_{n+2}H_n+6=2G_{n+1}(H_{n+1}+(-1)^{n+1})$.

Lemma 3: For every positive integer n, $G_nH_{n+1}+G_{n+1}H_n+3=\frac{1}{3}(G_{n+1}-G_n)(S_{n-1}+(-1)^{n+1})$.

Proof of Lemma 1: Using Proposition H(1), we see that

$$\begin{aligned}2H_nH_{n+1}+1&=2\left(\frac{(1+\sqrt{3})\gamma^{n-2}-(\sqrt{3}-1)\delta^{n-2}}{6}-\frac{(-1)^{n-2}}{2}\right)\\&\quad\times\left(\frac{(1+\sqrt{3})\gamma^{n-1}-(\sqrt{3}-1)\delta^{n-1}}{6}-\frac{(-1)^{n-2}}{2}\right)+1\\&=\frac{2}{36}((1+\sqrt{3})^2\gamma^{2n-3}+(\sqrt{3}-1)^2\delta^{2n-3}-2(\gamma+\delta))-\frac{2}{9}+1\\&\quad-\frac{2}{18}((\sqrt{3}+1)\gamma^{n-2}(\gamma-1)-(\sqrt{3}-1)\delta^{n-2}(\delta-1))(-1)^{n-2},\end{aligned}$$

since $\gamma\delta=1$. Now we know that $\gamma-1=\sqrt{3}+1$, $\delta-1=1-\sqrt{3}$, $\gamma+\delta=4$, $(1+\sqrt{3})^2=2\gamma$, and $(\sqrt{3}-1)^2=2\delta$. Hence,

$$\begin{aligned}2H_nH_{n+1}+1&=\frac{\gamma^{2n-2}}{9}+\frac{\delta^{2n-2}}{9}-\frac{4}{9}-\frac{2}{9}+1-\frac{(-1)^{n-2}2(\gamma^{n-1}+\delta^{n-1})}{9}\\&=\frac{(\gamma^{n+1}+\delta^{n+1})^2-2}{9}-\frac{6}{9}+1+\frac{2(-1)^{n-1}(\gamma^{n-1}+\delta^{n-1})}{9}\\&=\left(\frac{\gamma^{n-1}+\delta^{n-1}}{3}+\frac{(-1)^{n-1}}{3}\right)^2=\left(\frac{S_{n-1}+(-1)^{n-1}}{3}\right)^2,\end{aligned}$$

as claimed.

We note that defining $P_0=0$ is consistent with the Pell sequence recurrence and allows the proofs to go through in the case $n=1$.

Proof of the Main Result: For n, k positive integers, it suffices to show that

$$E(n,k)E(n,k+2)+1 \quad\text{and}\quad 2E(n,k)E(n,k+1)+1$$

are integer squares. The proof breaks into two parts: First, we expand $E(n,k)E(n,k+2)+1$ and find that

$$\begin{aligned}E(n,k)E(n,k+2)+1&=\frac{1}{4}(G_kP_{2n}-H_kP_{2n-2})(G_{k+2}P_{2n}-H_{k+2}P_{2n-2})+1\\&=\frac{1}{4}(G_kG_{k+2}P_{2n}^2+H_kH_{k+2}P_{2n-2}^2-P_{2n}P_{2n-2}(G_kH_{k+2}+G_{k+2}H_k))+1.\end{aligned}$$

Now, by Property 2,

$$2P_{2n}^2+1=\left(\frac{3P_{2n}-P_{2n-2}}{2}\right)^2$$

i.e.,

$$1=\frac{P_{2n}^2}{4}+\frac{P_{2n-2}^2}{4}-\frac{6}{4}P_{2n}P_{2n-2}.$$

Using Propositions G(2) and H(2), we find that in the expansion of $E(n,k)E(n,k+2)+1$ the coefficients of P_{2n}^2 and P_{2n-2}^2 are

$$\frac{G_{k+1}^2}{4} \quad\text{and}\quad \frac{(H_{k+1}-(-1)^{k+1})}{4},$$

respectively. By Lemma 2, we find that the coefficient of $P_{2n}P_{2n-2}$ is

$$-2G_{k+1}\frac{H_{k+1}+(-1)^{k+1}}{4}.$$

Hence,

$$E(n,k)E(n,k+2)+1=\frac{1}{4}(G_{k+1}P_{2n}-(H_{k+1}+(-1)^{k+1})P_{2n-2})^2,$$

which—since the Pell sequence subscripts are even—is an integer square, as desired. Next, we expand $E(n,k)E(n,k+1)+1$ and find that

$$\begin{aligned}E(n,k)E(n,k+1)+1&=\frac{1}{4}(2G_kP_{2n}-H_kP_{2n-2})(G_{k+1}P_{2n}-H_{k+1}P_{2n-2})+1\\&=\frac{1}{4}(2G_kG_{k+1}P_{2n}^2+2H_kH_{k+1}P_{2n-2}^2-2P_{2n}P_{2n-2}(G_kH_{k+1}+G_{k+1}H_k))+1.\end{aligned}$$

As before, recall that

$$1=\frac{P_{2n}^2}{4}+\frac{P_{2n-2}^2}{4}-\frac{6}{4}P_{2n}P_{2n-2}.$$

Using Proposition G(3) and Lemma 1, we find that in the expansion of $2E(n,k)E(n,k+1)+1$ the coefficients of P_{2n}^2 and P_{2n-2}^2 are

$$\frac{(G_{k+1}-G_k)^2}{4} \quad\text{and}\quad \frac{1}{4}\left(\frac{(S_{k-1}+(-1)^{k-1}}{3}\right)^2,$$

respectively. By Lemma 3, we find that the coefficient of $P_{2n}P_{2n-2}$ is

$$\frac{2}{4}\frac{(G_{k+1}-G_k)(S_{k-1}+(-1)^{k-1})}{3}.$$

Hence,

$$2E(n,k)E(n,k+1)+1=\frac{1}{4}\left((G_{k+1}-G_k)P_{2n}-\frac{S_{k-1}+(-1)^{k-1}}{3}P_{2n-2}\right)^2$$

which—by Proposition S(2) and the fact that the Pell sequence subscripts are even—is an integer square, as desired.

Diophantine Quadruples: Let us recall that a Diophantine quadruple is an ordered quadruple (a,b,c,d) of positive integers such that $ab+1$, $ac+1$, $ad+1$, $bc+1$, $bd+1$, and $cd+1$ are all integer squares. A recent result on Diophantine quadruples is the following (see [2]).

Theorem 5: If (a,b,c) is a Diophantine triplet for which $ab=1=x^2$, $ac+1=y^2$, $bc+1=z^2$, and $d=a+b+c+2abc+2xyz$, then (a,b,c,d) is a Diophantine quadruple.

This result and our Theorem 4 produce an infinite family of Diophantine quadruples, namely,

$$\begin{aligned}a&=E(n,k), & b&=2E(n,k+1),\\ c&=E(n,k+2), & d&=a+b+c+2abc+2xyz,\end{aligned}$$

where

$$x = \frac{1}{4}\left((G_{k+1} - G_k)P_{2n} - \frac{S_{k-1} + (-1)^{k+1}}{3} P_{2n-2} \right)^2,$$

$$y = \frac{1}{4}(G_{k+1}P_{2n} - (H_{k+1} + (-1)^{k+1})P_{2n-2})^2,$$

$$z = \frac{1}{4}\left((G_{k+2} - G_{k+1})P_{2n} - \frac{S_k + (-1)^k}{3} P_{2n-2} \right)^2.$$

For example, if we let $n = 4$ and $k = 3$, we obtain the Diophantine quadruple

$$(2990, 22428, 41796, 11211312362908);$$

the six relevant squares are p^2, q^2, r^2, s^2, t^2, and u^2, where (p, q, r, s, t, u) is the sextuple

$$(8189, 11179, 30617, 183089661, 501445225, 684534886).$$

APPENDIX

In this appendix, we give proofs of Lemmas 2 and 3.

Lemma 2: For every positive integer n, $G_nH_{n+2} + G_{n+2}H_n + 6 = 2G_{n+1}(H_{n+1} + (-1)^{n+1})$.

Proof: Let us abbreviate $G_nH_{n+2} + G_{n+2}H_n + 6$ by LHS. From Propositions G and H, we see that

$$\text{LHS} = \frac{\gamma^n - \delta^n}{\gamma - \delta} \frac{(1+\sqrt{3})\gamma^n - (\sqrt{3}-1)\delta^n}{6} - \frac{(-1)^{n-2}}{3}$$
$$+ \frac{\gamma^{n+2} - \delta^{n+2}}{\gamma - \delta} \frac{(1+\sqrt{3})\gamma^{n-2} - (\sqrt{3}-1)\delta^{n-2}}{6} - \frac{(-1)^{n-2}}{3} + 6.$$

Using the facts that $\gamma\delta = 1$, $(1+\sqrt{3})\delta^4 + (\sqrt{3}-1)\gamma^4 = 82\sqrt{3}$, $\gamma - \delta = 2\sqrt{3}$, and a little algebra, we find that

$$\text{LHS} = \frac{1}{6(\gamma - \delta)}\left(2(1+\sqrt{3})\gamma^{2n} + 2(\sqrt{3}-1)\delta^{2n} - 2\sqrt{3} - 8(-1)^n\gamma^{n+1} + 8(-1)^n\delta^{n+1} - 82\sqrt{3}\right) + 6$$
$$= \frac{2(1+\sqrt{3})\gamma^{2n} + 2(\sqrt{3}-1)\delta^{2n}}{6(\gamma - \delta)} - \frac{8}{6}G_{n+1}(-1)^n - 1.$$

On the other hand, abbreviating $2G_{n+1}(H_{n+1} + (-1)^{n+1})$ by RHS, we similarly see that

$$\text{RHS} = \frac{2}{6}\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta}((1+\sqrt{3})\gamma^{n-1} - (\sqrt{3}-1)\delta^{n-1} - 2(-1)^{n-1}) + 2(-1)^{n+1}G_{n+1}$$
$$= \frac{2((1+\sqrt{3})\gamma^{2n} + (\sqrt{3}-1)\delta^{2n} - (1+\sqrt{3})\delta^2 - (\sqrt{3}-1)\gamma^2)}{6(\gamma - \delta)} - \left(\frac{4}{6} - 2\right)(-1)^{n+1}G_{n+1}.$$

But $2((1+\sqrt{3})\delta^2 + (\sqrt{3}-1)\gamma^2) = 6(\gamma - \delta) = 12\sqrt{3}$ and $(-1)^n = -(-1)^{n+1}$, so

$$\text{RHS} = \frac{2(1+\sqrt{3})\gamma^{2n} + 2(\sqrt{3}-1)\delta^{2n}}{6(\gamma - \delta)} - 1 - \frac{8}{6}(-1)^n G_{n+1}.$$

Hence, $\text{LHS} = \text{RHS}$ and the lemma is proved.

Lemma 3: For every positive integer n, $G_nH_{n+1} + G_{n+1}H_n + 3 = \frac{1}{3}(G_{n+1} - G_n)(S_{n-1} + (-1)^{n+1})$.

Proof: Let us abbreviate $G_nH_{n+1}+G_{n+1}H_n+3$ by LHS. Using Propositions G and H and the equalities cited in the proof of Lemma 2, we see that

$$\begin{aligned}\text{LHS} &= \frac{\gamma^n-\delta^n}{\gamma-\delta}\,\frac{(1+\sqrt{3})\gamma^{n-1}-(\sqrt{3}-1)\delta^{n-1}-2(-1)^{n-1}}{6}\\ &+\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}\,\frac{(1+\sqrt{3})\gamma^{n-2}-(\sqrt{3}-1)\delta^{n-2}-2(-1)^{n-2}}{6}+3.\end{aligned}$$

A little more algebra leads to the equation:

$$\begin{aligned}\text{LHS} &= \frac{1}{6(\gamma-\delta)}(2(1+\sqrt{3})\gamma^{2n-1}+2(\sqrt{3}-1)\delta^{2n-1}-(\sqrt{3}-1)4\gamma^2\\ &-(\sqrt{3}+1)4\delta^2+2(-1)^{n-1}(\gamma^n(\gamma-1)-\delta^n(\delta-1)))+3.\end{aligned}$$

We use the facts that $(\sqrt{3}-1)4\gamma^2+(\sqrt{3}+1)4\delta^2=24\sqrt{3}$ and $6(\gamma-\delta)=12\sqrt{3}$ and arrive at

$$\text{LHS}=\frac{1}{6(\gamma-\delta)}(2(1+\sqrt{3})\gamma^{2n-1}+2(\sqrt{3}-1)\delta^{2n-1}+2(-1)^n((1+\sqrt{3})\gamma^n+(\sqrt{3}-1)\delta^n))+1.$$

On the other hand, abbreviating $\frac{1}{3}(G_{n+1}-G_n)(S_{n-1}+(-1)^{n-1})$ by RHS, we similarly see by Proposition S that

$$\begin{aligned}\text{RHS} &= \frac{1}{3(\gamma-\delta)}(\gamma^{n+1}-\delta^{n+1}-\gamma^n+\delta^n)(\gamma^{n-1}+\delta^{n-1}+(-1)^{n-1})\\ &= \frac{1}{3(\gamma-\delta)}(\gamma^{2n-1}(1+\sqrt{3})+\delta^{2n-1}(\sqrt{3}-1)+(1+\sqrt{3})\gamma\\ &+(\sqrt{3}-1)\delta+(-1)^{n-1}(\gamma^n(1+\sqrt{3})+\delta^n(\sqrt{3}-1))).\end{aligned}$$

Since $\gamma-1=1+\sqrt{3}$, $\delta-1=-(\sqrt{3}-1)$, and $(1+\sqrt{3})\gamma+(\sqrt{3}-1)\delta=6\sqrt{3}$, we find that

$$\text{RHS}=\frac{1}{3(\gamma-\delta)}((1+\sqrt{3})\gamma^{2n-1}+(\sqrt{3}-1)\delta^{2n-1}+(-1)^{n-1}((1+\sqrt{3})\gamma^n+(\sqrt{3}-1)\delta^n))+1.$$

Hence, LHS = RHS and this proves Lemma 3.

REFERENCES

1. M. N. Deshpande. Problem No. 10622. *Amer. Math. Monthly* **104** (1997):870.
2. A. Dujella. "Generalized Fibonacci Numbers and the Problem of Diophantus." *The Fibonacci Quarterly* **34.1** (1996):25-30.
3. A. Dujella. "A Problem of Diophantus and Pell Numbers." *Applications of Fibonacci Numbers* **7**:69-76. Ed. G. E. Bergum et al. Dordrecht: Kluwer, 1998.
4. V. E. Hoggatt, Jr., & G. E. Bergum. "A Problem of Fermat and Fibonacci Sequences." *The Fibonacci Quarterly* **15.4** (1977):323-30.

AMS Classification Numbers: 11B39, 11B61

❖❖❖