ALGORITHMIC DETERMINATION OF THE ENUMERATOR FOR SUMS OF THREE TRIANGULAR NUMBERS

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1. INTRODUCTION

In order to lend greater precision to statements of results and methods of proof, we begin our discussion with a definition.

Definition 1.1: As usual, $\mathbb{P} := \{1, 2, 3, \ldots\}$, $\mathbb{N} := \mathbb{P} \cup \{0\}$, and $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$. Then, for each $n \in \mathbb{N}$,

$$t_3(n) := \left| \left\{ (h, j, k) \in \mathbb{N}^3 \,|\, n = \frac{h(h+1)}{2} + \frac{j(j+1)}{2} + \frac{k(k+1)}{2} \right\} \right|;$$

and $q(n) :=$ the number of partitions of n into distinct parts. We define $q(0) := 1$ and $q(n) := 0$ for $n < 0$. The function $q(n)$, $n \in \mathbb{N}$, is generated by the infinite product expansion

$$\prod_1^\infty (1 + x^n) = \sum_0^\infty q(n) x^n,$$

which is valid for each complex number x such that $|x| < 1$.

As so many arithmetical discussions do, our discussion begins with Gauss, who first proved the following theorem. (The result was conjectured by Fermat about 150 years earlier.)

Theorem 1.2: Every natural number can be represented by a sum of three triangular numbers, i.e., for each $n \in \mathbb{N}$, $t_3(n) > 0$.

In this paper our major objective is to give an algorithmic procedure for computing $t_3(n)$, $n \in \mathbb{N}$. This is accomplished by the following two results.

Theorem 1.3: For each $n \in \mathbb{N}$,

$$q(n) + 2 \sum_{k \in \mathbb{P}} (-1)^k q(n - k^2) = \begin{cases} (-1)^m, & \text{if } n = m(3m \pm 1)/2, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Theorem 1.4: For each $n \in \mathbb{N}$,

$$t_3(n) = q(n) - \sum_{k \in \mathbb{P}} (-1)^k q(n - 3k^2 + 2k)(3k - 1) + \sum_{k \in \mathbb{P}} (-1)^k q(n - 3k^2 - 2k)(3k + 1). \tag{1.2}$$

For a proof of Theorem 1.3, see [1, pp. 1-2]. Section 2 is dedicated to the proof of Theorem 1.4.

2. PROOFS

In our development we require the following three identities:

$$\prod_1^\infty (1 + x^n)(1 - x^{2n-1}) = 1; \tag{2.1}$$

$$\prod_1^\infty \frac{1-x^{2n}}{1-x^{2n-1}} = \sum_0^\infty x^{n(n+1)/2}; \tag{2.2}$$

$$\prod_1^\infty \frac{(1-x^{2n})(1-a^2x^{2n-2})(1-a^{-2}x^{2n})}{(1+ax^{2n-1})(1+a^{-1}x^{2n-1})} = \sum_{-\infty}^\infty x^{n(3n+2)}(a^{-3n}-a^{3n+2}). \tag{2.3}$$

Identities (2.1) and (2.2) are valid for all complex numbers x such that $|x|<1$, while (2.3) is valid for each pair of complex numbers a, x such that $a \neq 0$ and $|x|<1$. For proofs of (2.1) and (2.2), see [2, pp. 277-84]; for a proof of (2.3), see [3, pp. 23-27]. In passing, we observe that the cube of the right-hand side of (2.2) generates the sequence $t_3(n)$, $n \in \mathbb{N}$. Proof of Theorem 1.4 is facilitated by the following lemma.

Lemma 2.1: For each complex number x such that $|x|<1$,

$$\prod_1^\infty \frac{(1-x^{2n})^3}{(1+x^{2n-1})^2} = \sum_{-\infty}^\infty (3n+1)x^{n(3n+2)}. \tag{2.4}$$

Proof: Multiply (2.3) by $-a^{-1}$ to get

$$(a-a^{-1})\prod_1^\infty \frac{(1-x^{2n})(1-a^2x^{2n})(1-a^{-2}x^{2n})}{(1+ax^{2n-1})(1+a^{-1}x^{2n-1})} = \sum_{-\infty}^\infty x^{n(3n+2)}(a^{3n+1}-a^{-3n-1}).$$

Now we operate on both sides of the foregoing identity with aD_a, D_a denoting differentiation with respect to a, subsequently, let $a \to 1$ and cancel a factor of 2 to draw the desired conclusion.

Returning to the proof of Theorem 1.4, we multiply both sides of (2.4) by

$$\prod_{n=1}^\infty (1+x^{2n-1})^{-1},$$

and appeal to (2.1), where we let $x \to -x$, to get

$$\begin{aligned}\sum_{n=0}^\infty (-1)^n t_3(n)x^n &= \prod_1^\infty \frac{(1-x^{2n})^3}{(1+x^{2n-1})^3} \\ &= \prod_{n=1}^\infty (1+(-x)^n)\sum_{-\infty}^\infty (3n+1)x^{n(3n+2)} \\ &= \sum_{n=0}^\infty (-1)^n q(n)x^n \sum_{-\infty}^\infty (3n+1)x^{n(3n+2)}.\end{aligned}$$

Now we expand the product of the two series and, subsequently, equate coefficients of like powers of x to prove Theorem 1.4.

Our algorithm proceeds in two steps:

(i) Use the recursive determination of q in Theorem 1.3 to compile a table of values of q, as in Table 1.

(ii) Utilizing Theorem 1.4 and the values of q computed in Table 1, we then compile a list of values of t_3, as shown in Table 2.

TABLE 1

n	$q(n)$	n	$q(n)$
0	1	13	18
2	1	14	22
3	2	16	32
4	2	17	38
5	3	18	46
6	4	19	54
7	5	20	64
8	6	21	76
9	8	22	89
10	10	23	104
11	12	24	122
12	15	25	142

TABLE 2

n	$t_3(n)$	n	$t_3(n)$
0	1	10	9
1	3	11	6
2	3	12	9
3	4	13	9
4	6	14	6
5	3	15	6
6	6	16	15
7	9	17	9
8	3	18	7
9	7	19	12

3. CONCLUDING REMARKS

The brief tables above are compiled to show the effectiveness of the algorithm. For a fixed but arbitrary choice of $n \in \mathbb{P}$, we observe that: (1) to compute $q(n)$ we need about $\sqrt{n}$ of the values $q(k)$, $0 \le k < n$; and then (2) to compute $t_3(n)$ we need $q(n)$ and about $\sqrt{4n/3}$ of the values $q(k)$, $0 \le k < n$. Doubtless, the formulas (1.1) and (1.2) can be adapted to machine computation, and the corresponding tables can then be extended indefinitely.

For given $n \in \mathbb{P}$, there are formulas that express $t_3(n)$ in terms of certain divisor functions. But, for each divisor function f, evaluation of $f(k)$, $k \in \mathbb{P}$, requires factorization of k. By comparison we observe that our algorithm is entirely additive in character. In a word, no factorization is required.

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